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(2, 2) - Total Domination in Graphs

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Abstract

A set S of vertices in a graph $G=(V, E)$ is a (2, 2)-total dominating set of G if every vertex in V is adjacent to at least 2 vertices in S and at least 2 vertices in $V-S$. The minimum cardinality of a (2, 2)-total dominating set is called the (2, 2)-total domination number of G and is denoted by $\gamma_{t2,2}(G)$. In this paper, we initiate a study of (2, 2)-total domination in graphs. Some bounds on $\gamma_{t2,2}(G)$ are found and its exact values for some standard graphs are obtained.

Keywords: total domination, 2-total domination, (2,2)-total domination.

Mathematics Subject Classification: 050C.

1. Introduction

The graphs considered here are finite, undirected without loops and multiple edges. Unless and otherwise stated, the graphs $G=(V, E)$ considered here have $p=|V|$ vertices and $q=|E|$ edges. For graph theoretic terminology, we refer to Harary[2].

A set $D \subseteq V$ is a dominating set of G if every vertex in $V-D$ is adjacent to some vertex in D. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. Recently many new domination parameters are given in the books by Kulli[3,6,7].

A set D of vertices in a graph G is a total dominating set if every vertex in V is adjacent to some vertex in D. The total domination number $\gamma_t(G)$ of G is the minimum cardinality of a total dominating set of G,[1]. This concept was generalized in Kulli[4] as follows:

A set $D \subseteq V$ is an n-total dominating set of G if every vertex in V is adjacent to at least n vertices in D. The n-total domination number $\gamma_{tn}(G)$ of G is the minimum cardinality of an n-total dominating set of G. If n=2, then $\gamma_{tn}(G)= \gamma_{t2}(G)$, see[5].

Kulli and Janakiram[8] introduced the concept of (n, m)-total domination as follows:

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A set $D \subseteq V$ of a graph G is an (n,m)-total dominating set of G if every vertex in V is adjacent to at least n vertices in D and at least m vertices in $V-D$. The (n,m)-total domination number $\gamma_{m,m}(G)$ of G is the minimum cardinality of an (n, m)-total dominating set of G.

2. Results

Definition 1. A set S of vertices in a graph $G=(V,E)$ is a (2,2)-total dominating set of G if every vertex in V is adjacent to at least 2 vertices in S and at least 2 vertices in $V-S$. The minimum cardinality of a (2,2)-total dominating set is called the (2,2)-total domination number of G and is denoted by $\gamma_{t2,2}(G)$.

Definition 2. A (2, 2)-total dominating set S is called a minimal (2, 2)-total dominating set if no proper subset of S is a (2, 2)-total dominating set.

We note that not all graphs have a (2, 2)-total dominating set.

Proposition 3. For any tree T, $\gamma_{t2,2}(T)$ does not exist.

Proposition 4. For any cycle C_p, $\gamma_{t2,2}(C_p)$ does not exist.

Proposition 5. For any wheel W_p, $\gamma_{t2,2}(W_p)$ does not exist.

Proposition 6. For any graph G with $\gamma_{t2,2}$-set,

$$\gamma_{t2}(G) \leq \gamma_{t2,2}(G). \qquad (1)$$

Proof: Clearly every (2,2)-total dominating set is a 2-total dominating set. Thus (1) holds.

We need the following results.

Proposition A[5]. For any complete graph K_p with $p\geq 3$ verices,

$$\gamma_{t2}(K_p) = 3.$$

Proposition B[5]. For any complete bipartite graph with $2\leq m\leq n$,

$$\gamma_{t2}(K_{m,n}) = 4.$$

We now obtain (2,2)-total domination number of some standard graphs.

Proposition 7. For any complete graph K_p with $p\geq 6$ vertices,

$$\gamma_{t2,2}(K_p) = 3. \qquad (2)$$

Proof: Let D be a γ_{t2}-set of K_p. Then by Proposition A, $|D|=3$. Since $\delta(K_p)\geq 5$, every vertex v in V is adjacent to at least 2 vertices in $V-D$. Thus D is a (2,2)-total dominating set of K_p and hence (2) follows from (1).

Proposition 8. For any complete bipartite graph $K_{m,n}$ with $4 \leq m \leq n$,

$$\gamma_{t2,2}(K_{m,n}) = 4. \qquad (3)$$

Proof: Let D be a γ_{t2} - set of $K_{m,n}$. Then by Proposition B, $|D|=4$. Since $\delta(K_{m,n}) \geq 4$, every vertex v in V is adjacent to at least 2 vertices in $V-D$.

Thus D is a (2, 2)-total dominating set of $K_{m,n}$ and hence (3) follows from (1).

Theorem C[5]. If G is a connected graph with $\delta(G) \geq 2$,

$$\gamma_t(G)+1 \leq \gamma_{t2}(G).$$

Theorem 9. If G is a connected graph with $\delta(G) \geq 2$, then

$$\gamma_t(G) + 1 \leq \gamma_{t2,2}(G) \qquad (4)$$

and this bound is sharp.

Proof: By Theorem 6, $\gamma_{t2}(G) \leq \gamma_{t2,2}(G)$ and by Theorem C, $\gamma_t(G) + 1 \leq \gamma_{t2}(G)$. Thus

$$\gamma_t(G) + 1 \leq \gamma_{t2,2}(G).$$

The complete graphs K_p with $p \geq 6$ vertices achieve this bound.

The following result gives a characterization of (2,2)-total minimal dominating sets.

Theorem 10. A (2,2)-total dominating set D of G is minimal if and only if for each vertex v in D, there exists a vertex u in $N(v)$ such that $|N(u) \cap D| = 2$.

Proof: Suppose D is a (2,2)-total minimal dominating set. On the contrary, suppose there exists a vertex v in D such that v does not satisfy the given condition. Then $D - \{v\}$ is not a (2, 2)-total dominating set of G, which is a contradiction. Thus v satisfies the given condition.

Converse is obvious.

A γ_{t2}-set is a minimum 2-total dominating set. Similarly a $\gamma_{t2,2}$–set can be defined.

We obtain a sufficient condition on γ_{t2}-set of G which is also $\gamma_{t2,2}$-set of G.

Theorem 11. If $\gamma_{t2}(G) \leq \delta(G) - 2$, then any γ_{t2}-set of G is a $\gamma_{t2,2}$-set.

Proof: This follows from the fact that each vertex in V is adjacent to at least 2 vertices in D and 2 vertices in $V - D$, where D is a γ_{t2}-set of G.

We now obtain a lower bound on $\gamma_{t2,2}(G)$.

Theorem 12. If G is a 4-regular graph with $\gamma_{t2,2}$-set, then

$$\left\lceil \frac{p}{3} \right\rceil \leq \gamma_{t_{2,2}}(G). \qquad (5)$$

Proof: Let D be a $\gamma_{t2,2}$-set of a 4-regular graph G. Since each vertex in V is adjacent to exactly 2 vertices in $V - D$, we have

$$2\gamma_{t2,2}(G) + \gamma_{t2,2}(G) \geq p.$$

Thus (5) holds.

We obtain an upper bound for $\gamma_{t2,2}(G)$.

Theorem 13. If a $\gamma_{t2,2}$-set exists in a graph G, then

$$\gamma_{t2,2}(G) \leq \frac{p}{2}. \qquad (6)$$

Proof: Since the complement of a (2, 2)-total dominating set of G is a (2, 2)-total dominating set, we have

$$\gamma_{t2,2}(G) - \gamma_{t2,2}(G) \leq p.$$

Thus (6) holds.

Problem 14. Characterize graphs G for which $\gamma_{t2,2}(G) = \gamma_t(G)+1$.

Problem 15. Characterize graphs G for which $\gamma_{t2}(G) = \gamma_{t2,2}(G)$.

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