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β_2 near-rings

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Abstract

In this paper we introduce the notion of β_2 near-rings and study some of their properties. We furnish a complete characterization and also a structure theorem for such near rings.

Keywords : β_2 near-ring, near-field.

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1 Introduction

A right near – ring$(N,+,\ .)$ is an algebraic system with two binary operations '+'and '·' suchthat
(1) $(N,+)$ is a group (with 0 as its identity element)
(2)$(N,\cdot)$ is a semi–group and (3) $(n_1n_2)n_3=n_1n_3n_2n_3$ for alln_1,n_2,n_3 N.

Throughout this paper, N stands for a right near – ring with at least two elements. Obviously, $0n=0$ for all n in N. N is said to be zero – symmetric if $n0=0$ for all n in N. As in[2], a sub group of $(M,+)$ of $(N,+)$ is called (i) a left N– subgroup if $MN\subseteq M$ (ii) an N– subgroup of N if $NM\subseteq M$ and an invariant N– subgroup of N if, M satisfies both (i) and (ii). Also in[2], N is defined to beweak commutative if $xyz=xzy$ for all x, y, z in N and N is said to have property P_4 if for all ideals I of N, $xy\in I$ implies $yx\in I$ for all x,y in N. The concept of a mate function in$N$ has been introduced in[4] with a viewto handling the regularity structurewith-considerable ease. A map 'f' from N into N is called (i) a mate function for N if $x=xf(x)x$(ii)a P_3 mate function if, in addition, $xf(x)=f(x)x$ for all x in N. Also the existence of matefunction is preserved underhomomorphisms.

Basic concepts and terms used but left undefined in this paper can be found in Pilz[2].

2 *Notations:*

(i) E denotes the set ofall idempotentsof N. (e in N is called an idempotent if $e^2=e$)

(ii) L denotes the set of all nilpotents of N. (a in N is nilpotent if $a^k=0$ for some positive integer k.)

(iii) $N_d=\{n\in N/n(x+y)=nx+ny$ for all x, y in $N\}$ – set of all distributive elements of N.

(iv) $C(N)=\{n\in N/ nx=xn$ for all x in $N\}$–centre of N.

(v) $N_0=\{n\in N/n0=0\}$–zero symmetric part of N.

3 Preliminary Results:

We freely make use of the following results and designate them as **R(1), R(2)**…etc.

R(1) N is subdirectly irreducible if and only if the intersection of any family of non-zero ideals of N is again non-zero.(Theorem 1.60, p.25 of [2])

R(2) N has no non-zero nilpotent elements if and only if $x^2=0\Rightarrow x=0$ for all x in N. (problem 14, p.9 of [3])

R(3) If f is a mate function for N, then for every x in N, $xf(x)$, $f(x)x\in E$ and $Nx=Nf(x)x$, $xN=xf(x)N$. (Lemma 3.2 of [4]).

R(4) A zero–symmetric near–ring N is a near–field if $N_d\neq\{0\}$ and for all $n\in N\text{-}\{0\}$, $Nn=N$. (Theorem 8.3, p.249 of [2]).

R(5) If $L=\{0\}$ and $N=N_0$ then (i) $xy=0\Rightarrow yx=0$ for all x,y in N. (ii) N has Insertion of Factors property – IFP for short – (i.e) for x,y in N, $xy=0\Rightarrow xny=0$ for all n in N (If N satisfies both (i) and (ii) then N is said to have (∗,IFP)) (Lemma 2.3 of [4]).

R(6) N has strong IFP if for all ideals I of N, $ab\in I\Rightarrow anb\in I$ for all a,b,n in N. (Proposition 9.2, p.289 of [2])

4 Definition of β_2 Near–Rings and Examples:

In this section we define β_2 near-rings and give certain examples of this new concept.

Definition 4.1: Let N be a right near-ring. If for every x,y in N, $xNy=xyN$ then we say that N is a β_2 near-ring.

Example 4.2: (a) Any constant near-ring is obviously a β_2 near-ring.

(b) The near-ring $(N,+,\cdot)$ where $(N,+)$ is the Klein's four group with $N=\{0,a,b,c\}$ where '·' is defined as per scheme 13, p. 408 of Pilz [2]

·	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	0	0	0
c	0	a	b	c

is not a β_2 near-ring since $aNb\neq abN$.

(c) We consider the near – ring $(N,+,\cdot)$ defined on the Klein's four group $(N,+)$ with $N=\{0, a, b, c\}$ where '·' is defined as per scheme 12, p.408 of Pilz [2].

·	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	0	0	0
c	0	a	0	a

This is a zero symmetric β_2 near-ring with no identity.

5. Properties of β_2 Near-Rings:

In this section, we prove some of the important properties of β_2 near – rings and give a complete characterization of such near-rings. We also obtain a structure theorem for β_2 near-ring.

Proposition 5.1 Let N be a β_2 near-ring. If N has identity 1, then N is zero-symmetric.

Proof : Let N be a β_2 near-ring. Then for all x, y in N, $xNy=xyN$. Putting $x=1$, we get $1Ny=1yN$ for all y in N, when $y=0$, $N0=0N=\{0\}$. It follows that N is zero-symmetric.

Remark 5.2 The converse of proposition 5.1 is not valid. For example, the near-ring cited in Example 4.2(c) is a zero-symmetric β_2 near-ring, but it has no identity.

Proposition 5.3 If N is a β_2 near-ring then $xNx=x^2N$ for all x in N.

Proof: When N is a β_2 near – ring, by definition, for all x,y in N, $xNy=xyN$.....(1). The resultfollows by replacing y by x in (1).

Remark 5.4 The converse of proposition 5.3 is not true. For example, the near – ring $(N,+,\cdot)$ where $(N,)$ is the Klein's four group {0,a,b,c} and '·' is defined as per scheme 20 p. 408, Pilz[2]

·	0	a	b	c
0	0	0	0	0
a	a	a	a	a
b	0	a	b	c
c	a	0	c	b

satisfies the condition $xNx=x^2N$ for all x in N. But itis not a β_2 near – ring. [sincebNabaN]

Proposition 5.5 Everyweakcommutative near – ringis β_2 near-ring.

Proof: Let N be a weak commutative near-ring....(1). Let $x,y\in N$. If $a\in xNy$, then there exists $n\in N$ such that $a=xny=xyn$[by (1)]. Therefore $a\in xyN$. Thus $xNy\subseteq xyN$......(2). On the other hand, if $b\in xyN$, then for some $n'\in N$, $b=xyn'=xn'y$ [by(1)]. Consequently, $xyN\subseteq xNy$ ....(3). Combination of (2) and (3) yields the desired result.

Proposition 5.6 Every β_2 near – ring has strong IFP.

Proof: Let N be a β_2 near – ring. Then $xNy=xyN$ for all x,y in N......(1). Let I be an ideal of N......(2) and let $ab\in I$. Now, for any $n\in N$, $anb\in aNb=abN$ [by(1)]$\subseteq IN\subseteq I$ [by(2)]. Now R(6), guarantees that N has strong IFP.

Theorem 5.7 Let N bea β_2 near-ring with identity 1. Then

(i) Every N-subgroup is invariant.

(ii) If $N=N_d$ then every left N-subgroup is an N-subgroup.

Proof: Since N is a β_2 near-ring, we have $xNy=xyN$ for all x,y in N.......(1)

(i) Let S be any N-subgroup of N. Then $S=\sum_{x\in s} Nx$....(2) Now, $NxN=N1xN=N1Nx$ [by(1)]$\subseteq Nx$....(3). Therefore $SN=(\sum_{x\in s} Nx)N$ [by(2)] $\subseteq \sum_{x\in s} NxN \subseteq \sum_{x\in s} Nx$ [by(3)]$=S$[by(2)]. Consequently,S is invariant.

(ii) Let S be any left N-subgroup of N. Then $S=\sum_{x\in s} xN$ ....(4). Now, $NxN=1NxN=1xNN$ [by(1)]$\subseteq xN$.......(5). Therefore $NS=N(\sum_{x\in s} xN) \subseteq \sum_{x\in s} NxN \subseteq \sum_{x\in s} xN$ [by(5)] =

S[by(4)]. Consequently, S is an N-subgroup.

Proposition 5.8 Homomorphic image of a β_2 near – ring is also a β_2 near – ring.

Proof : Straight forward.

Theorem 5.9 Every β_2 near –ring N is isomorphic to a subdirect product of subdirectly irreducible β_2 near-rings

Proof: By Theorem 1.62, p.26 of Pilz[2], N isisomorphic to a subdirect product of subdirectly irreducible near-rings $N'_i s$ and each N_i is a homomorphic image of N under the projection map π_i. The rest of the proof is taken care of by Proposition 5.8.

6 β_2 Near-Rings which admit mate functions:

Tobegin with we shall prove thenecessary and sufficient condition for a β_2 near-ring to admit mate functions.

Lemma 6.1 Let N be a β_2 near-ring. Then N admits a mate function if and only if $x \in x^2N$ for all x in N.

Proof: We first observe from proposition 5.3 that, as N is β_2, $xNx=x^2N$ for all x in N.............(1) For the 'only if' part, let f be a mate function for N. Then for all x in N, $x=xf(x)x \in xNx$. It follows that $x \in x^2N$ [by(1)]. For the 'if' part, let $x \in x^2N$ for all x in N. Appealing to(1) we get, $x=xnx$ for some n in N. By setting $n=f(x)$, we see that f is a mate function for N.

Henceforth we assume that N admits mate functions.

Theorem 6.2 Let N be a zero-symmetric β_2 near - ring with a mate function 'f'. Then we have,

(i) $L=\{0\}$.
(ii) N has $(*, IFP)$
(iii) $E \subseteq C(N)$ if $E \subseteq N_d$.
(iv) Any ideal of N is completely semi prime.
(v) N has property P_4.

Proof: **(i)** Since f is a mate function for N, Lemma 6.1 demands that $x \in x^2N$ for all x in N. Therefore $x = x^2n$ for some n in N. Suppose $x^2 = 0$. Clearly, then $x = 0$. Now, R(2) guarantees that $L = \{0\}$.

(ii) By (i) $L = \{0\}$. Now, R(5) guarantees that N has $(*, IFP)$.

(iii) Let $e \in E$. Since N is β_2, $eNe=e.eN= eN$. Therefore for any n in N, $ene = eu$ and $en = eve$ for some u, v in N. Now, $ene = eue$and $ene = eve$. Thus $ene = en$ for all n in N..........(1). We also have, $e(ne - ene) = 0$ [since $E \subseteq N_d$]$\Rightarrow ene(ne - ene) = 0$ [by(ii)]. And $ne(ne - ene) = n.0 = 0$. Consequently, $(ne - ene)^2 = 0$ and (i) guarantees $ne - ene = 0$. Therefore, $ene = ne$ for all n in N......(2). From (1) and (2), we get $en = ne$ for all n in N. Thus $E \subseteq C(N)$.

(iv) Let I be an ideal of N and let $a^2 \in I$ for a in N. Now, since f is a matefunction for N, for all a in N. $a = af(a)a \in aNa$. Since N is β_2, $a \in a^2 N$. Therefore for some n in N. $a=a^2 n \in IN \subseteq I$. Consequently, I is completely semiprime.

(v) Let I be an ideal of N and let $xy \in I$. Since

N is zero-symmetric, $NI \subseteq I$......(3) and $IN \subseteq I$......(4). Now, $(yx)^2 = yxyx = y(xy)x \in NIN = (NI)N \subseteq IN$ [by (3)] $\subseteq I$ [by (4)] (i.e) $(yx)^2 \in I$. Appealing to (iv) we get $yx \in I$. Consequently, N has property P_4.

Theorem 6.3 If N is a β_2 near-ring with a mate function then every N-subgroup of N is also so.

Proof: Let N be a β_2 near-ring with a mate function f and let A be an N-subgroup of N. Then $NA \subseteq A$........(1). By Theorem 5.7 (i), $AN \subseteq A$.....(2). Let $a, b \in A$. Then for $c \in A$, $acb \in aAb \subseteq aNb = abN$. [since N is β_2]. Therefore, for some n in N, $acb = abn = abf(b)bn \in abf(b)(AN) \subseteq abf(b)A$ [by(2)] $\subseteq abNA \subseteq abA$ [by(1)]. Thus aAb⊆abA.....(3) On the other hand, if $a, b \in A$, then for $x \in A$, $abx \in abA \subseteq abN = aNb$ [since N is β_2]. Hence, for some $n_1 \in N$, $abx = an_1 b = an_1bf(b)b \in aNAf(b)b \subseteq aAf(b)b$[by(1)]$\subseteq aANb \subseteq aAb$ [by(2)]. Thus $abA \subseteq aAb$.....(4) From (3) and (4), $aAb = abA$ for all a, b in A and hence A is a β_2 near-ring. We observe that the function g defined on A by $g(x) = f(x)xf(x)$ serves as a mate function for A. Thiscompletes the proof of the theorem.

We furnish below a characterization for β_2 near-rings with mate functions

Theorem 6.4 Let $N = N_d$ be a near-ring with a mate function 'f'. Then N is β_2 if and only if $xN = x^2N$ for all x in N and $E \subseteq C(N)$.

Proof: For the 'only if' part, first we observe that $E \subseteq C(N)$......(1) follows from Theorem 6.2 (iii). Now for any x in N, obviously $x^2N \subseteq xN$......(2). On the other hand, if $b \in xN$, then for some n in N, $b = xn = (xf(x)x)n = xnf(x)x$ [by (1)] $\in xNx = x^2N$ [by Proposition 5.3]. Consequently, xNx^2N........(3). Combining (2) and (3), $xN = x^2N$ for all x in N. For the if part, first we show that f is a P_3 mate function. For any $x \in N$ we have $x = xf(x)x = f(x)x^2$ [since $E \subseteq C(N)$] $\Rightarrow (xf(x) - f(x)x)x = 0 \Rightarrow x(xf(x) - f(x)x) = 0$ [by Theorem 6.2 (ii)]$\Rightarrow xf(x)(xf(x) - f(x)x) = 0$ [by Theorem 6.2 (ii)] and $f(x)x(xf(x) - f(x)x) = f(x).0 = 0$. [since $N = N_0$]. Consequently, $(xf(x) - f(x)x)^2 = 0$ and hence $xf(x) = f(x)x$ [by R(2)].........(4). Hence f is a P_3 mate function. Now, $xyN = xyf(y)N = xNyf(y)$ [since $E \subseteq C(N)$] $= xNf(y)y$ [by(4)] $= xNy$ [by R(3)]. Thus N is a β_2 near-ring.

With a view to establishing astructure theorem we prove the following theorem.

Theorem 6.5 Let $N = N_d$ be a β_2 near-ring with a mate function f. Then N is subdirectly irreducible if and only if N is a near-field.

Proof: For the 'only if' part, first we shall show that no non-zero idempotent of N is zero-divisor. Let J be the set of all non – zeroidempotentsin N which are zero-divisors. Let $I = \cap\{(0 : e)/e \in J\}$. Since N is subdirectly irreducible, R(1) demands that $I \neq \{0\}$. Let $a \in I - \{0\}$. Then $ae = 0$ for all e in J..........(1). By Theorem 6.2 (ii), $ea = 0 \Rightarrow ef(a)a = 0 \Rightarrow f(a)a \in J$. Therefore, $af(a)a = 0$ [by (1)]$\Rightarrow a = 0$ which is a contradiction to $a \neq 0$. Consequently, no non – zero idempotent of$N$isazero – divisor......(2). Let M be any non-zero N–subgroup of $N$and $0 \neq x \in M$. Theorem 6.4 demands that $xN = x^2N$ for all x in N. Therefore, for any n in N, $xn = x^2 n_1$ for some n_1 in N.

Then $x(n-xn_1)=0$ [since $N=N_d$]$\Rightarrow xf(x)(n-xn_1)=0$ [by Theorem 6.2 (ii)]$\Rightarrow n-xn_1=0$ [by(2)]$\Rightarrow n=xn_1\in MN\subseteq M$ [by Theorem 5.7]. Thus $N\subseteq M$. Consequently, N has no non – trivial N-subgroups..........(3). Let $n\in N-\{0\}$. Then by (3), $Nn=N$. Further since $N_d\neq\{0\}$, it follows that N is a near-field [by R(4)]. The proof of 'if' partis obvious.

We conclude our discussionby proving the following structure theorem.

Theorem 6.6 Let $N=N_d$ be a β_2 near-ring with a mate function f. Then N is isomorphic to a subdirect product of near-fields.

Proof: By Theorem 5.9, N is isomorphic to a subdirectproduct ofsubdirectly irreducible β_2 near-rings. Since N has a mate function it follows that each N_i also has a matefunction. The rest of the proof is taken care of by Theorem 6.5.

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