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Fixed Point Theorems for four Mappings on Probabilistic Metric Spaces

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Abstract

Our main goal to proved common fixed point theorems for four self-mappings with φ-contractive condition in a complete Menger spaces. In this paper we used the concept and definitions some know results[1-12].

1. Preliminaries

Fixed Point Theory is a beautiful mixture of pure and applied mathematics, topology and geometry.

The probabilistic metric spaces is an important part of Stochastic Analysis, to develop the fixed point theory in such spaces. Fixed point theory in probabilistic metric spaces can be considered as a part of Probabilistic Analysis, which is a very dynamic area of mathematical research. There are many results in fixed point theory in probablisitc metric space. Metric spaces were introduced by Gahler in 1964, and since then there have been many fixed point theorems proved in metric spaces and as a generalization of metric spaces.

The idea of introducing probabilistic notions into geometry was one of the great thoughts of Karl Menger. His motivation came from the idea that positions, distances, areas, volumes, etc., all are subject to variation in measurement in practice. In 1942, Menger[6] published a note entitled Statistical Metrics. The first result from the fixed point theory in Probabilistic metric spaces was obtained by Sehgal and Bharucha – Reid in[9]. Schweizer and Sklar[8] took up the work, initiated by Menger[6] and developed what is now called the theory of probabilistic metric spaces[12]. There are many results in fixed point theory in probabilistic metric space., since then there have been many fixed point theorems proved in metric spaces and as a generalization of metric spaces, Fixed point theory in Menger spaces is a developed branch of mathematics.

In this paper, we will prove two common fixed point theorems for four self-mappings with φ-contractive condition in a Menger space, which generalize some results of Dedei´ and Sarapa[4,5], and Sehgal and Bharucha-Reid[9].

1.1 Definition: A self mapping F: R → R is said to be a distribution if it is non-decreasing left continuous with inf $\{F(t) : t \in R\} = 0$ and $\sup\{F(t) : t \in R\} = 1$.

We shall denote by the set of all distribution functions while G will always denote the specific distribution function defined by

$$G(t) = \begin{cases} 0, & t \leq 0, \\ 1, & t > 0. \end{cases} \qquad (1.1)$$

1.2 Definition: A probabilistic metric space is an ordered pair (X, $\mathfrak{F}$) consisting of a nonempty set X and a mapping $\mathfrak{F}$ from X × X into the collections of all distribution functions on R. For x, y ∈ X, we denote the distribution function $\mathfrak{F}(x, y)$ by $F_{x,y}$ and $F_{x,y}(u)$ represents the value of (x, y) at u ∈ R. The functions $F_{x,y}$ are assumed to satisfy the following conditions:[1,2]

(1) $F_{x,y}(u) = 1$ for all u > 0 if and only if x = y,
(2) $F_{x,y}(0) = 0$ for all x, y in X,
(3) $F_{x,y}(u) = F_{y,x}(u)$ for all x, y in X, and
(4) if $F_{x,y}(u) = 1$ and $F_{y,z}(v) = 1$, then $F_{x,z}(u + v) = 1$ for all x, y, z in X and u, v > 0.

1.3 Definition: A mapping t : [0, 1] × [0, 1] → [0, 1] is called a t-norm if
(1) t(a, 1) = a, t(0, 0) = 0,
(2) t(a, b) = t(b, a),
(3) t(c, d) ≥ t(a, b) for c ≥ a, d ≥ b, and
(4) t(t(a, b), c) = t(a, t(b, c)).

1.3 Definition: A Menger space is a triplet (X, $\mathfrak{F}$, t), where (X, $\mathfrak{F}$) is a PM-space, t is a T-norm, and the generalized triangle inequality

$$F_{x,z}(u + v) \geq t(F_{x,y}(u), F_{y,z}(v)) \qquad (1.2)$$

The concept of neighborhoods in a Menger space was introduced by Schweizer and Sklar[8].

holds for all x, y, z in X and u, v > 0.

1.4 Definition: Let (X, $\mathfrak{F}$, t) be a Menger space. If x ∈ X, ε > 0, and λ ∈ (0, 1), then an (ε, λ)-neighborhood of x, called Ux (ε, λ), is defined by

$$U_x(\varepsilon, \lambda) = \{y \in X : F_{x,y}(\varepsilon) > 1 - \lambda\}. \qquad (1.3)$$

An (ε, λ)-topology in X is the topology induced by the family
$\{U_x(\varepsilon, \lambda): x \in X, \varepsilon > 0, \lambda \in (0, 1)\}$ of neighborhood.

1.5 Remark: If t is continuous, then Menger space (X, $\mathfrak{F}$, t) is a Hausdorû space in the (ε, λ)-topology. (see[8]).

1.6 Definition: Let (X, $\mathfrak{F}$, t) be a complete Menger space and A ⊂ X. Then A is called a bounded set if

$$\lim_{u\to\infty} \inf_{x,y\in A} F_{x,y}(u)=1 \qquad (1.4)$$

We denoted B(X) as the family of

nonempty bounded subsets of a complete Menger space X.

For all A, B $\in$ B(X) and for all u >0, we define

$${}_{\delta}F_{A,B}(u) = \inf\{F_{x,y}(u) : x \in A, y \in B\}$$

$${}_{D}F_{A,B}(u) = \sup\{F_{x,y}(u) : x \in A, y \in B\}$$

$${}_{H}F_{A,B}(u) = \inf\left\{\sup_{a\in A}\inf_{b\in B} F_{a,b}(u),\ \sup_{b\in B}\inf_{a\in A} F_{a,b}(u)\right\} \quad (1.5)$$

1.7 Remark: It is clear that ${}_{\delta}F_{A,B}(u) = {}_{\delta}F_{B,A}(u)$,

${}_{\delta}F_{\{x\},B}(u) = {}_{\delta}F_{x,B}(u)$, ${}_{D}F_{\{x\},B}(u) = {}_{D}F_{x,B}(u)$ *and* ${}_{H}F_{\{x\},B}(u) = {}_{H}F_{x,B}(u)$, and

${}_{H}F_{A,B}(u) = {}_{H}F_{B,A}(u)$ for all A,B$\in$B(X) and u>0.

If A = {x}, we denote

${}_{\delta}F_{\{x\},B}(u) = {}_{\delta}F_{x,B}(u)$, ${}_{D}F_{\{x\},B}(u) = {}_{D}F_{x,B}(u)$

and ${}_{H}F_{\{x\},B}(u) = {}_{H}F_{x,B}(u)$

1.8 Definition: Let (X, $\mathfrak{F}$, t) be a complete Menger space, and let T : X $\to$ B(X) be a set-valued function and I : X $\to$ X a single-valued function. Then we say that S and I are compatible if

$$\lim_{n\to\infty} {}_{H}F_{SIx_n, ISx_n}(u) = 1 \quad (1.6)$$

whenever {x_n} is a sequence in X such that

$$\lim_{n\to\infty} F_{Ix_n, Sx_n}(u) = 1 \quad \forall u > 0. \quad (1.7)$$

1.9 Definition: Let {A_n} be a sequence in B(X). We say that {A_n} δ-converges to a set A in X if

$$\min\left\{{}_{\delta}F_{A,B}(u),\ {}_{\delta}F_{B,C}(v)\right\} \le \min\left\{F_{a,b}(u),\ F_{b,c}(v)\right\} \le F_{a,c}(u+v)\ . \quad (1.8)$$

and it is denoted by $A_n \xrightarrow{\delta} A$.

1.10 Definition: Let F_1, $F_2 \in \mathfrak{I}$. The algebraic sum $F_1 \oplus F_2$ of F_1 and F_2 is defined by

$$(F_1 \oplus F_2)(t) = \sup_{t_1+t_2=t} \min\{F_1(t_1),\ F_2(t_2)\}$$

Obviously

$(F_1 \oplus F_2)(2t) \ge \min\{F_1(t_1), F_2(t_2)\}$ for all t $\ge$ 0

2. Main results

Let R^+ denote the set of all nonnegative real numbers and N denote the set of all positive integers, and let (X, $\mathfrak{F}$, t) be a Menger space with t(x, y) = min(x, y).

Before prove main results we proof some lemmas.

2.1 Lemma: Let (X, $\mathfrak{F}$, min) be a Menger space. Then for A, B, C $\in$ B(X) and for u, v > 0,

$${}_{\delta}F_{A,C}(u+v) \ge \min\left\{{}_{\delta}F_{A,B}(u),\ {}_{\delta}F_{B,C}(v)\right\}. \quad (2.1)$$

Proof. For all u, v > 0, we have

$$\min\left\{{}_{\delta}F_{A,B}(u),\ {}_{\delta}F_{B,C}(v)\right\} \le \min\left\{F_{a,b}(u),\ F_{b,c}(v)\right\} \le F_{a,c}(u+v) \quad (2.2)$$

for each a $\in$ A, b $\in$ B, and c $\in$ C.

This implies that $\min\left\{{}_{\delta}F_{A,B}(u),\ {}_{\delta}F_{B,C}(v)\right\} \le {}_{\delta}F_{A,C}(u+v)$.

2.2. Lemma: Let (X, $\mathfrak{F}$, min) be a Menger space. Then for A, B ∈ B(X), c ∈ X, and for u, v > 0,

$$_H F_{A,C}(u+v) \geq \min\{{}_H F_{A,B}(u),\ {}_H F_{B,C}(v)\}. \tag{2.3}$$

Proof. Since for each a, b, c ∈ X and for all u, v > 0,

$$F_{a,c}(u+v) \geq \min\{F_{a,b}(u),\ F_{b,c}(v)\}. \tag{2.4}$$

By taking inf $_{c\in C}$, we have

$$\inf_{c\in C} F_{a,c}(u+v) \geq \min\{F_{a,b}(u),\ \inf_{c\in C} F_{b,c}(v)\}. \tag{2.5}$$

Hence,

$$\sup_{a\in A}\inf_{c\in C} F_{a,c}(u+v) \geq \sup_{a\in A}\min\{F_{a,b}(u),\ \inf_{c\in C} F_{b,c}(v)\}.$$

$$= \min\{\sup_{a\in A} F_{a,b}(u),\ \inf_{c\in C} F_{b,c}(v)\}. \tag{2.6}$$

$$\leq \min\{\sup_{a\in A}\inf_{b\in B} F_{a,b}(u),\ \inf_{c\in C} F_{b,c}(v)\}.$$

Next, by taking $\sup_{b\in B}$ we have

$$\sup_{a\in A}\inf_{c\in C} F_{a,c}(u+v) \geq \sup_{b\in B}\min\{\sup_{a\in A}\inf_{b\in B} F_{a,b}(u),\ \inf_{c\in C} F_{b,c}(v)\}$$

$$\geq \min\{\sup_{a\in A}\inf_{b\in B} F_{a,b}(u),\ \sup_{b\in B}\inf_{c\in C} F_{b,c}(v)\} \tag{2.7}$$

Similarly, for each a, b, c∈X and for all u, v>o,

$$F_{a,c}(u+v) \geq \min\{F_{a,b}(u), F_{b,c}(v)\}. \tag{2.8}$$

By taking inf c∈C, we have

$$\inf_{a\in A} F_{a,c}(u+v) \geq \min\left\{\inf_{a\in A} F_{a,b}(u), F_{b,c}(v)\right\} \tag{2.9}$$

Hence,

$$\sup_{c\in C}\inf_{a\in A} F_{a,c}(u+v) \geq \sup_{c\in C}\min\left\{\inf_{a\in A} F_{a,b}(u), F_{b,c}(v)\right\}$$

$$=\min\left\{\inf_{a\in A} F_{a,b}(u), \sup_{c\in C} F_{b,c}(v)\right\}$$

$$\geq \min\left\{\inf_{a\in A} F_{a,b}(u), \sup_{c\in C, b\in B}\inf F_{b,c}(v)\right\}$$

Next, by taking $\sup_{b\in B}$, we have

$$\sup_{c\in C}\inf_{a\in A} F_{a,c}(u+v) \geq \sup_{b\in B}\min\left\{\inf_{a\in A} F_{a,b}(u), \sup_{c\in C}\inf_{b\in B} F_{b,c}(v)\right\}$$

$$= \min\left\{\sup_{b\in B}\inf_{a\in A} F_{a,b}(u), \sup_{c\in C}\inf_{b\in B} F_{b,c}(v)\right\} \tag{2.11}$$

Therefore, we obtain that

$$_H F_{A,C}(u+v) = \min\left\{\sup_{c\in C}\inf_{a\in A} F_{a,c}(u+v), \sup_{a\in A}\inf_{c\in C} F_{a,c}(u+v)\right\}$$

$$\geq \min\left\{\sup_{b\in B}\inf_{a\in A} F_{a,b}(u), \sup_{c\in C}\inf_{b\in B} F_{a,c}(v), \sup_{a\in A}\inf_{b\in B} F_{a,b}(u), \sup_{b\in B}\inf_{c\in C} F_{a,c}(v)\right\} \tag{2.12}$$

$$= \min\{{}_H F_{A,B}(u), {}_H F_{A,C}(v)\}.$$

2.3. Lemma: Let (X, $\mathfrak{F}$, min) be a Menger space. If A, B ∈ B(X), then $\lim_{u\to\infty} {}_\delta F_{A,B}(u) = 1$.

Proof. For any x ∈ A and y ∈ B, by Lemma 2.1, we have

$$_\delta F_{A,B}(u+v) \geq \min\left\{{}_\delta F_{A,x}\left(\frac{u}{3}\right), {}_\delta F_{x,y}\left(\frac{u}{3}\right), {}_\delta F_{y,B}\left(\frac{u}{3}\right)\right\} \tag{2.13}$$

Letting u →∞, we have

$$\lim_{u\to\infty} {}_{\delta}F_{A,B}\left(u\right) \geq \min\left\{\lim_{u\to\infty} {}_{\delta}F_{A,x}\left(\frac{u}{3}\right),\right.$$

$$\left.\lim_{u\to\infty} {}_{\delta}F_{x,y}\left(\frac{u}{3}\right), \lim_{u\to\infty} {}_{\delta}F_{y,B}\left(\frac{u}{3}\right)\right\} \quad (2.14)$$

Since x ∈ A, y ∈ B, and A, B ∈ B(X), we have

$$\lim_{u\to\infty} {}_{\delta}F_{A,x}\left(\frac{u}{3}\right) = 1 \quad (2.15)$$

Similarly, we have

$$\lim_{u\to\infty} {}_{\delta}F_{y,B}\left(\frac{u}{3}\right) = 1 \quad (2.16)$$

By the definition of the PM-space, we have that $\lim_{u\to\infty} F_{x,y}(u/3) = 1$.
Therefore, we conclude that

$$\lim_{u\to\infty} {}_{\delta}F_{A,B}\left(u\right) = 1 \quad (2.17)$$

This completes the proof.

The following lemma which was introduced by Chang[3], will play an important role for this result.

2.4. Lemma: If $\varphi : R^+ \to R^+$ is a strictly increasing, continuous function such that $0<\varphi(u)<u$ for all u >0, $\lim_{u\to\infty} \varphi(u)=\infty$, and if for each u>0, $\varphi^0(u)=u$ and $\varphi^{-n}(u)=\varphi^{-1}(\varphi^{-n+1}(u))$ for each n∈N are denoted, then $\lim_{n\to\infty} \varphi^{-n}(u)=\infty$.

In the sequel, we let
$\Phi = \{\varphi : R^+ \to R^+ : \varphi$ is a strictly increasing, continuous function with $\varphi(t) < t$ for all $t > 0\}$.

2.5. Lemma: Let (X, 𝔉, min) be a Menger space and $\{Y_n\}$ a sequence in B(X). If for each u > 0 and for each n ∈ N,

$${}_{\delta}F_{Y_{n+1},Y_{n+2}}\left(\phi(u)\right) \geq {}_{\delta}F_{Y_n,Y_{n+1}}\left(u\right), \ \varphi\in\Phi, \quad (2.18)$$

then
$$\lim_{n\to\infty} {}_{\delta}F_{Y_n,Y_{n+1}}\left(u\right) = 1 \quad (2.19)$$

Proof. For u > 0, by induction, we have

$${}_{\delta}F_{Y_{n+1},Y_{n+2}}\left(u\right) \geq {}_{\delta}F_{Y_n,Y_{n+1}}\left(\phi^{-1}(u)\right) \geq \ldots \geq {}_{\delta}F_{Y_1,Y_2}\left(\phi^{-n}(u)\right),$$

for each n ∈ N. (2.20)

By Lemma 2.4, we also have that $\varphi^{-n}(u) \to\infty$, as n →∞.
Next, since Y_n is a bounded set and ${}_{\delta}F_{Y_1,Y_2}\left(\phi^{-n}(u)\right), \to 1$ as n→∞, hence we have

$$\lim_{n\to\infty} {}_{\delta}F_{Y_{n+1},Y_{n+2}}\left(u\right) = 1 \quad (2.21)$$

2.6. Lemma: Let (X, 𝔉, min) be a Menger space, and let A, B ∈ B(X). If

$${}_{\delta}F_{A,B}\left(\phi(u)\right) \geq {}_{\delta}F_{A,B}\left(u\right) \text{ for } u > 0, \quad (2.22)$$

then A = B = a, for some a ∈ X.

Proof. For u > 0, by induction, we have

$${}_{\delta}F_{A,B}\left(u\right) \geq {}_{\delta}F_{A,B}\left(\phi^{-1}(u)\right) \geq \ldots \geq {}_{\delta}F_{A,B}\left(\phi^{-n}(u)\right) \quad (2.23)$$

Since A,B ∈ B(X), by Lemma 2.3, we have

$$\lim_{n\to\infty} {}_{\delta}F_{A,B}\left(\phi^{-n}(u)\right) = 1 \quad (2.24)$$

and by Lemma 2.5, we have $\lim_{n\to\infty} {}_{\delta}F_{A,B}\left(u\right) = 1$ for u > 0. Thus we conclude that A = B = {a} for some a ∈ X.

The following lemma was introduced by Schweizer and Sklar[8].

2.7. Lemma: Let (X, 𝔉,min) be a

Menger space. If $a_n \to a$ and $b_n \to b$, then for u > 0,

$$\lim_{n\to\infty} {}_\delta F_{A,B}\left(\phi^{-n}(u)\right) = 1 \quad (2.25)$$

From Lemma 2.7, we conclude the following lemma.

2.8. Lemma: Let (X, $\mathfrak{F}$, min) be a Menger space. If $A_n \xrightarrow{\delta} a$ and $B_n \xrightarrow{\delta} b$, then for u > 0,

$$\lim_{n\to\infty} {}_\delta F_{A_n,B_n}(u) = F_{a,b}(u) \quad (2.26)$$

Proof. For u > 0 and for ε > 0. Since $F_{a,b}$ (u) is left continuous function at u, there exists a positive number k with 0 < 2k < u such that $F_{a,b}(u) - F_{a,b}(u-2k) < \varepsilon$.

Since k > 0 and $A_n \xrightarrow{\delta} a$, $B_n \xrightarrow{\delta} b$ hence we may take m ∈ N such that for n ≥ m,

$${}_\delta F_{A_n,a}(k) \ge F_{a,b}(u-2k),$$

$${}_\delta F_{B_n,b}(k) \ge F_{a,b}(u-2k). \quad (2.27)$$

Hence, for n > m,

$$\begin{aligned} {}_\delta F_{A_n,B_n}(u) &\ge \min\{{}_\delta F_{A_n,b}(u-k), {}_\delta F_{b,B_n}(k)\} \\ &\ge \min\{{}_\delta F_{A_n,b}(k), {}_\delta F_{a,b}(u-k), \\ &\quad {}_\delta F_{b,B_n}(k)\} = F_{a,b}(u-k), \end{aligned} \quad (2.28)$$

and hence

$${}_{-\delta} F_{A_n,B_n}(u) \le -F_{a,b}(u-2k). \quad (2.29)$$

Therefore, we conclude that

$$F_{a,b}(u) - {}_\delta F_{A_n,B_n}(u) < F_{a,b}(u) - F_{a,b}(u-2k) < \varepsilon. \quad (2.30)$$

Taking $\lim_{n\to\infty}$ inf, we have

$$F_{a,b}(u) - \lim_{n\to\infty} {}_\delta F_{A_n,B_n}(u) < \varepsilon \quad (2.31)$$

For any $a_n \in A_n$, $b_n \in B_n$, since $A_n \xrightarrow{\delta} a$ and $B_n \xrightarrow{\delta} b$, we have $a_n \to a$, $b_n \to b$. Thus, for u>0

$${}_\delta F_{A_n,B_n}(u) < F_{a_n,b_n}(u). \quad (2.32)$$

Taking limn→∞ inf, we have

$$\liminf_{n\to\infty} {}_\delta F_{A_n,B_n}(u) < \liminf_{n\to\infty} F_{a_n,b_n}(u). \quad (2.33)$$

By Lemma 2.7, we have

$$\liminf_{n\to\infty} F_{a_n,b_n}(u) = F_{a,b}(u), \text{ and so } F_{a,b}(u)$$

$$-\liminf_{n\to\infty} {}_\delta F_{A_n,B_n}(u) \ge 0. \quad (2.34)$$

Therefore, for any ε > 0,

$$\varepsilon > F_{a,b}(u) - \liminf_{n\to\infty} {}_\delta F_{A_n,B_n}(u) \ge 0. \quad (2.35)$$

This implies that

$$\liminf_{n\to\infty} {}_\delta F_{A_n,B_n}(u) = F_{a,b}(u) \quad (2.36)$$

The following two theorems are our main results for this paper.

Theorem 2.9. Let (X, $\mathcal{F}$, min) be a complete Menger space. Let f , g, η, ξ : X→X be four single - valued functions, and let S,T: X → B(X) two set-valued functions. If the following conditions are satisfied:[7]

(i) S(X) ⊂ ξg(X), T(X) ⊂ η f (X),

(ii) $\eta f = f\eta$, $\xi g = g\xi$, S f = f S, Tg = gT,

(iii) η f or ξg is continuous,

(iv) (S, ηf) and (T, ξg) are compatible, and

(v) for u > 0,

$${}_\delta F_{Sx,Ty}\varphi(u)$$
$$\ge \min\{F_{\eta fx,\xi gy}(u), {}_\delta F_{\xi gy,Ty}(u), {}_\delta F_{\xi gy,Sx}(\beta u),$$
$$[F_{\eta fx,gy} \oplus F_{gy,Ty}](2-\beta)(u),$$

$[F_{Sx,\eta fx} \oplus F_{Ty,\xi gy}](2-\beta)(u)$, $2.\ F_{Sx,\xi gy}(u)$.

$$\left[\frac{1+F_{Ty,\xi gy}(u)}{1+F_{Sy,\xi gy}(u)}\right]\} \qquad (2.37)$$

for all x, y ∈ X, β ∈ (0,2), where φ ∈ Φ, then f , g, η, ξ, S, and T have a unique common fixed point z in X.

Proof. Let $x_0 \in X$. Define the sequence $\{x_n\}$ recursively as follows:

$$\xi g x_{2n+1} \in Sx_{2n} = z_{2n}\,,$$

$$\eta f x_{2n+2} \in Tx_{2n+1} = z_{2n+1}\,. \qquad (2.38)$$

For n ∈ N and for all u > 0, and β = (1 − α) with α ∈ (0, 1),

$$
\begin{aligned}
&{}_\delta F_{z_{2n},z_{2n+1}}(\phi(u))\\
&\quad = {}_\delta F_{Sx_{2n},Tx_{2n+1}}(\phi(u))\\
&\quad \geq \min\{F_{\eta fx_{2n},\xi gx_{2n+1}}(u), {}_\delta F_{\xi gx_{2n+1},Tx_{2n+1}}(u), {}_\delta F_{\xi gx_{2n+1},Sx_{2n}}(\beta u),\\
&\qquad [F_{\eta fx_{2n},\xi gx_{2n+1}} \oplus F_{\xi gx_{2n+1},Tx_{2n+1}}](2-\beta)(u),\ [F_{Sx_{2n},\eta fx_{2n}} \oplus F_{Tx_{2n+1},\xi gx_{2n+1}}](2-\beta)(u),\\
&\qquad 2.\ F_{Sx_{2n},\xi gx_{2n+1}}(u).\left[\frac{1+F_{Tx_{2n+1},\xi gx_{2n+1}}(u)}{1+F_{Sx_{2n},\xi gx_{2n+1}}(u)}\right]\}\\
&\quad \geq \min\{{}_\delta F_{z_{2n-1},z_{2n}}(u),\ {}_\delta F_{z_{2n},z_{2n+1}}(u), {}_\delta F_{z_{2n},z_{2n}}((1-\alpha)u),\\
&\qquad [F_{z_{2n-1},z_{2n}} \oplus F_{z_{2n},z_{2n+1}}](\alpha+1)(u),\ [F_{z_{2n},z_{2n-1}} \oplus F_{z_{2n+1},z_{2n}}](1+\alpha)(u),\\
&\qquad 2.\ F_{z_{2n},z_{2n}}(u).\left[\frac{1+F_{z_{2n+1},z_{2n}}(u)}{1+F_{z_{2n},z_{2n}}(u)}\right]\}\\
&\quad \geq \min\{{}_\delta F_{z_{2n-1},z_{2n}}(u),\ {}_\delta F_{z_{2n},z_{2n+1}}(u), 1,\\
&\qquad [F_{z_{2n-1},z_{2n}} \oplus F_{z_{2n},z_{2n+1}}](\alpha+1)(u),\ [F_{z_{2n},z_{2n-1}} \oplus F_{z_{2n+1},z_{2n}}](1+\alpha)(u),\\
&\qquad 2.\ F_{z_{2n},z_{2n}}(u).\left[\frac{1+F_{z_{2n+1},z_{2n}}(u)}{2}\right]\}\\
&\quad = \min\{{}_\delta F_{z_{2n-1},z_{2n}}(u),\ {}_\delta F_{z_{2n},z_{2n+1}}(u),\\
&\qquad [F_{z_{2n-1},z_{2n}} \oplus F_{z_{2n},z_{2n+1}}](\alpha+1)(u)\}
\end{aligned}
$$

As t – norms it is continuous, letting $\alpha \to 1$, and **definition 2.5** we have

$$= min\{{}_\delta F_{z_{2n-1}, z_{2n}}(u),\ {}_\delta F_{z_{2n}, z_{2n+1}}(u),$$
$$[F_{z_{2n-1}, z_{2n}} \oplus F_{z_{2n}, z_{2n+1}}](2)(u)\}$$

$${}_\delta F_{z_{2n}, z_{2n+1}}(\phi(u)) = min\{{}_\delta F_{z_{2n-1}, z_{2n}}(u),\ {}_\delta F_{z_{2n}, z_{2n+1}}(u)\} \tag{2.40}$$

By Lemma 2.6, we have

$${}_\delta F_{z_{2n}, z_{2n+1}}(\phi(u)) \geq {}_\delta F_{z_{2n-1}, z_{2n}}(u) \tag{2.41}$$

Similarly, we also can prove that for $n \in N$ and for all $u > 0$,

$${}_\delta F_{z_{2n+1}, z_{2n+2}}(\phi(u)) \geq {}_\delta F_{z_{2n}, z_{2n+1}}(u) \tag{2.42}$$

So, we have

$${}_\delta F_{z_{n+1}, z_{n+2}}(\phi(u)) \geq {}_\delta F_{z_n, z_{n+1}}(u) \tag{2.43}$$

By Lemma 2.5, we conclude that

$$\lim_{n\to\infty} {}_\delta F_{z_n, z_{n+1}}(\phi(u)) = 1, \quad \forall\ u > 0 \tag{$*$}$$

Now, we consider the condition (v) with $\beta = 1$, and then we claim that

for $\varepsilon > 0$, $\lambda \in (0,1)$ there is $M(\varepsilon, \lambda) \in N$ such that ${}_\delta F_{z_n, z_m}(\varepsilon) = 1 - \lambda$ for $n, m \geq M$. (2.44)

If it is not the case, then there exists $\varepsilon > 0$, $\lambda \in (0, 1)$ such that for $k \in N$, there exist $n_k > m_k \in k$ such that

(1) n_k is even and m_k is odd,

(2) ${}_\delta F_{z_{n_k}, z_{m_k}}(\varepsilon') < 1 - \lambda'$, and

(3) n_k is the smallest even number such that (1) and (2) hold.

By ($*$), we may choose $m_1 \in N$ such that for $n \geq m_1$,

$${}_\delta F_{z_n, z_{n+1}}(\min\{\frac{\varepsilon'}{2}, \frac{\phi^{-1}(\varepsilon') - \varepsilon'}{2}\}) < 1 - \lambda' \tag{2.45}$$

So for $k > m_1$, $n_k \geq m_{k+3}$, and so for $k > m_1$,

$$1 - \lambda' > {}_\delta F_{z_{n_k}, z_{m_k}}(\varepsilon') = {}_\delta F_{Sx_{n_k}, Tx_{m_k}}(\varepsilon')$$

$$\geq min\{F_{\eta fx_{n_k}, \xi gx_{m_k}}(\phi^{-1}(\varepsilon')), {}_\delta F_{\xi gx_{m_k}, Tx_{m_k}}(\phi^{-1}(\varepsilon')), {}_\delta F_{\xi gx_{m_k}, Sx_{n_k}}(\phi^{-1}(\varepsilon')),$$
$$[F_{\eta fx_{n_k}, \xi gx_{m_k}} \oplus F_{\xi gx_{m_k}, Tx_{m_k}}](\phi^{-1}(\varepsilon')),\ [F_{Sx_{n_k}, \eta fx_{n_k}} \oplus F_{Tx_{m_k}, \xi gx_{m_k}}](\phi^{-1}(\varepsilon')),$$
$$2.\ F_{Sx_{n_k}, \xi gx_{m_k}}(\phi^{-1}(\varepsilon')).\ \left[\frac{1 + F_{Tx_{m_k}, \xi gx_{m_k}}(\phi^{-1}(\varepsilon'))}{1 + F_{Sx_{n_k}, \xi gx_{m_k}}(\phi^{-1}(\varepsilon'))}\right]\}$$

$$\geq min\{F_{z_{n_{k-1}}, z_{m_{k-1}}}(\phi^{-1}(\varepsilon')), {}_\delta F_{z_{m_{k-1}}, z_{m_k}}(\phi^{-1}(\varepsilon')), {}_\delta F_{z_{m_{k-1}}, z_{n_k}}(\phi^{-1}(\varepsilon')),$$
$$[F_{z_{n_{k-1}}, x_{m_{k-1}}} \oplus F_{z_{m_{k-1}}, z_{m_k}}](\phi^{-1}(\varepsilon')),\ [F_{z_{n_k}, z_{n_{k-1}}} \oplus F_{z_{m_k}, z_{m_{k-1}}}](\phi^{-1}(\varepsilon')),$$
$$2.\ F_{z_{n_k}, z_{m_{k-1}}}(\phi^{-1}(\varepsilon')).\ \left[\frac{1 + F_{z_{m_k}, z_{m_{k-1}}}(\phi^{-1}(\varepsilon'))}{1 + F_{z_{n_k}, z_{m_{k-1}}}(\phi^{-1}(\varepsilon'))}\right]\}$$

$$
{}_{\delta}F_{z_{m_{k-1}},z_{n_k}}\left(\phi^{-1}(\varepsilon')\right) \geq min\{{}_{\delta}F_{z_{m_{k-1}},z_{n_{k-1}}}\left(\frac{\phi^{-1}(\varepsilon')+\varepsilon'}{2}\right), {}_{\delta}F_{z_{n_{k-1}},z_{n_k}}\left(\frac{\phi^{-1}(\varepsilon')-\varepsilon'}{2}\right)\}
$$

$$
\geq min\{{}_{\delta}F_{z_{n_{k-1}},z_{n_{k-2}}}\left(\frac{\phi^{-1}(\varepsilon')-\varepsilon'}{2}\right), {}_{\delta}F_{z_{n_{k-2}},z_{m_{k-1}}}(\varepsilon'), {}_{\delta}F_{z_{n_{k-1}},z_{n_k}}\left(\frac{\phi^{-1}(\varepsilon')-\varepsilon'}{2}\right)\} \tag{2.46}
$$

Since using (2.1)

$$
{}_{\delta}F_{z_{n_{k-1}},z_{m_k}}\left(\phi^{-1}(\varepsilon')\right) \geq min\{{}_{\delta}F_{z_{n_{k-1}},z_{n_{k-2}}}\left(\phi^{-1}(\varepsilon')\right), {}_{\delta}F_{z_{n_{k-2}},z_{m_k}}\left(\phi^{-1}(\varepsilon')\right)\}
$$

$$
\begin{aligned}
&\geq min\{F_{z_{n_{k-1}},z_{m_{k-1}}}\left(\phi^{-1}(\varepsilon')\right), {}_{\delta}F_{z_{m_{k-1}},z_{m_k}}\left(\phi^{-1}(\varepsilon')\right), {}_{\delta}F_{z_{m_{k-1}},z_{n_k}}\left(\phi^{-1}(\varepsilon')\right),\\
&\quad [F_{z_{n_{k-1}},x_{m_{k-1}}} \oplus F_{z_{m_{k-1}},z_{m_k}}](\phi^{-1}(\varepsilon')), \; [F_{z_{n_k},z_{n_{k-1}}} \oplus F_{z_{m_k},z_{m_{k-1}}}](\phi^{-1}(\varepsilon')),\\
&\quad 2.\, F_{z_{n_k},z_{m_{k-1}}}\left(\phi^{-1}(\varepsilon')\right).\left[\frac{1+F_{z_{m_k},z_{m_{k-1}}}(\phi^{-1}(\varepsilon'))}{1+F_{z_{n_k},z_{m_{k-1}}}(\phi^{-1}(\varepsilon'))}\right]\}
\end{aligned}
$$

$$
{}_{\delta}F_{z_{m_{k-1}},z_{n_k}}\left(\phi^{-1}(\varepsilon')\right) \geq min\{{}_{\delta}F_{z_{m_{k-1}},z_{n_{k-1}}}\left(\frac{\phi^{-1}(\varepsilon')+\varepsilon'}{2}\right), {}_{\delta}F_{z_{n_{k-1}},z_{n_k}}\left(\frac{\phi^{-1}(\varepsilon')-\varepsilon'}{2}\right)\}
$$

$$
\geq min\{{}_{\delta}F_{z_{n_{k-1}},z_{n_{k-2}}}\left(\frac{\phi^{-1}(\varepsilon')-\varepsilon'}{2}\right), {}_{\delta}F_{z_{n_{k-2}},z_{m_{k-1}}}(\varepsilon'), {}_{\delta}F_{z_{n_{k-1}},z_{n_k}}\left(\frac{\phi^{-1}(\varepsilon')-\varepsilon'}{2}\right)\} \tag{2.47}
$$

so for $k > m_1$, we have

$$
1-\lambda' > {}_{\delta}F_{z_{n_k},z_{m_k}}(\varepsilon') \geq 1-\lambda', \tag{2.48}
$$

which is a contradiction. And, since X is complete, hence for any choice of z_n in Z_n, the sequence $\{z_n\}$ must converge to some point, say, z in X. The point z is independent of the choice of z_n and so we have

$\eta f x_{2n} \to z, \xi g x_{2n+1} \to z$, $Sx_{2n} \to \{z\}$, $Tx_{2n+1} \to \{z\}$. (2.49)

That is, for $u \to 0$, (2.50)

Assume that the function ηf is continuous, then for u > 0, we have

$$
\lim_{n\to\infty} F_{(\eta f)^2 x_{2n},\eta fz}(u) \to 1, \qquad \lim_{n\to\infty} {}_{\delta}F_{\eta fSx_{2n},\eta fz}(u) = 1 \tag{2.51}
$$

By $\lim_{n\to\infty} F_{\eta fx_{2n},z}(u) = 1$ and $\lim_{n\to\infty} {}_{\delta}F_{Sx_{2n},z}(u) = 1$, we obtain $\lim_{n\to\infty} {}_{\delta}F_{Sx_{2n},\eta fx_{2n}}(u) = 1$. Since S and ηf are compatible, and for $u>0$, $\lim_{n\to\infty} {}_{\delta}F_{Sx_{2n},\eta fx_{2n}}(u) = 1$, we have

$\lim_{n\to\infty} {}_{H}F_{\eta fSx_{2n},S\eta fx_{2n}}(u) = 1$ and $\lim_{n\to\infty} {}_{H}F_{\eta fSx_{2n},\eta fz}(u)$

$$
\geq \min\{{}_{H}F_{\eta fSx_{2n},S\eta fx_{2n}}(\frac{u}{2}), {}_{H}F_{\eta fSx_{2n},\eta fz}(\frac{u}{2}).
$$

And, since

$$\lim_{n\to\infty} {}_H F_{\eta f Sx_{2n}, S\eta fx_{2n}}(\frac{u}{2}) = 1, \quad \lim_{n\to\infty} {}_H F_{\eta f Sx_{2n}, \eta fz}(\frac{u}{2}) = 1$$

we have

$$\lim_{n\to\infty} {}_H F_{S\eta fx_{2n}, \eta fz}(u) = \lim_{n\to\infty} {}_\delta F_{S\eta fx_{2n}, \eta fz}(u) = 1 \quad (2.52)$$

In order to complete the proof, we will divide it into 5 steps as follows:

Step 1. For $u > 0$ with β=1 in the condition (v),

$$\begin{aligned} &{}_\delta F_{S\eta fx_{2n}, Tx_{2n+1}}(\phi(u)) \\ &\geq min\{F_{(\eta f)^2 x_{2n}, \xi gx_{2n+1}}(u), {}_\delta F_{\xi gx_{2n+1}, Tx_{2n+1}}(u), {}_\delta F_{\xi gx_{2n+1}, S\eta fx_{2n}}(\beta u), \\ &\quad [F_{(\eta f)^2 x_{2n}, \xi gx_{2n+1}} \oplus F_{\xi gx_{2n+1}, Tx_{2n+1}}](2-\beta)(u), [F_{S\eta fx_{2n}, (\eta f)^2 x_{2n}} \oplus F_{Tx_{2n+1}, \xi gx_{2n+1}}](2-\beta)(u), \\ &\quad 2.\, F_{S\eta fx_{2n}, \xi gx_{2n+1}}(u). \left[\frac{1 + F_{Tx_{2n+1}, \xi gx_{2n+1}}(u)}{1 + F_{S\eta fx_{2n}, \xi gx_{2n+1}}(u)}\right]\} \end{aligned} \quad (2.53)$$

Taking $\lim_{n\to\infty}$, by Lemma 2.8,

$$\begin{aligned} &F_{\eta fz, z}(\phi(u)) \\ &\geq min\{F_{\eta fz, z}(u), F_{z, z}(u), F_{z, \eta fz}(\beta u), \\ &\quad [F_{\eta fz, z} \oplus F_{z, z}](2-\beta)(u), [F_{\eta fz, \eta fz} \oplus F_{z, z}](2-\beta)(u), \\ &\quad 2.\, F_{\eta fz, z}(u). \left[\frac{1 + F_{z, z}(u)}{1 + F_{\eta fz, z}(u)}\right]\} \end{aligned} \quad (2.54)$$

So we get $\eta f z = z$.

Step 2. For u > 0 with β = 1 in the condition (v),

$${}_\delta F_{Sz, z}(\varphi(u))$$

$$\begin{aligned} &= \lim_{n\to\infty} \inf {}_\delta F_{Sz, Tx_{2n}}(\phi(u)) \\ &\geq \lim_{n\to\infty} \inf min\{F_{\eta fz, \xi gx_{2n+1}}(u), {}_\delta F_{\xi gx_{2n+1}, Tx_{2n+1}}(u), {}_\delta F_{\xi gx_{2n+1}, Sz}(\beta u), \\ &\quad [F_{\eta fz, \xi gx_{2n+1}} \oplus F_{\xi gx_{2n+1}, Tx_{2n+1}}](2-\beta)(u), [F_{Sz, \eta fz} \oplus F_{Tx_{2n+1}, \xi gx_{2n+1}}](2-\beta)(u), \\ &\quad 2.\, F_{Sz, \xi gx_{2n+1}}(u). \left[\frac{1 + F_{Tx_{2n+1}, \xi gx_{2n+1}}(u)}{1 + F_{Sz, \xi gx_{2n+1}}(u)}\right]\} \end{aligned}$$

$$\geq min\{F_{\eta fz},{}_{z}(u),{}_{\delta} F_{z},{}_{z}(u),{}_{\delta} F_{z},{}_{Sz}(\beta u),$$
$$[F_{z},{}_{z} \oplus F_{z,z}](2-\beta)(u), [F_{Sz},{}_{z} \oplus F_{z,z}](2-\beta)(u),$$
$$2.\ F_{Sz},{}_{z}(u).\left[\frac{1+F_{z,z}(u)}{1+F_{Sz,z}(u)}\right]\} \tag{2.55}$$

So we get Sz = {z}.
Hence, by Steps 1 and 2, we have Sz = {z} = {ηfz}.
Step 3. By the condition (i), since SX ⊂ ξgX, there exists z' ∈ X such that {ξgz'} = Sz ={z}.

So for any u > 0 with β = 1 in the condition (v)

$${}_{\delta}F_{Sx_{2n},Tz'}(\phi(u))$$

$$\geq min\{F_{\eta fx_{2n}},{}_{\xi gz'}(u),{}_{\delta} F_{\xi gz'},{}_{Tz'}(u),{}_{\delta} F_{\xi gz'},{}_{\eta fx_{2n}}(\beta u),$$
$$[F_{\eta fx_{2n}},{}_{\xi gz'} \oplus F_{\xi gz',Tz'}](2-\beta)(u), [F_{\eta fz'},{}_{\eta fx_{2n}} \oplus F_{z',\xi gz'}](2-\beta)(u),$$
$$2.\ F_{\eta fz'},{}_{\xi gz'}(u).\left[\frac{1+F_{z',\xi gz'}(u)}{1+F_{\eta fz',\xi gz'}(u)}\right]\} \tag{2.56}$$

Taking $\lim_{n\to\infty}$ inf, by Lemma 2.8,

$${}_{\delta}F_{z,Tz'}(\phi(u))$$
$$\geq min\{F_{z},{}_{z}(u),{}_{\delta} F_{z},{}_{Tz'}(u),F_{z},{}_{z}(\beta u),$$
$$[F_{z,z} \oplus F_{z,Tz'}](2-\beta)(u), [F_{z},{}_{z} \oplus F_{z,z}]$$
$$(2-\beta)(u), \tag{2.57}$$

So we get Tz'={z}. Hence, {ξgz'}={z} = Tz'.

By Step 2, we may let {z} = {ηf z} = {Sz} = {ξgz'} = {Tz'}.

Since S and ηf are compatible and {ηfz} = Sz, we get ηfSz = Sηfz, that is, {ηfz} = Sz.

Now,

$$\geq min\{F_{\eta fz},{}_{\xi gz'}(u),{}_{\delta} F_{\xi gz'},{}_{Tz'}(u),{}_{\delta} F_{\xi gz'},{}_{Sz}(\beta u),$$
$$[F_{\eta fz},{}_{\xi gz'} \oplus F_{\xi gz',Tz'}](2-\beta)(u), [F_{Sz},{}_{\eta fz}$$
$$\oplus F_{z',\xi gz'}](2-\beta)(u),$$
$$2.\ F_{\eta fz'},{}_{\xi gz'}(u).\left[\frac{1+F_{z',\xi gz'}(u)}{1+F_{Sz,\xi gz'}(u)}\right]\}$$
$$= {}_{\delta}F_{\eta fz,z}(u) = {}_{\delta}F_{Sz,z}(u). \tag{2.58}$$

This implies Sz = {z} = {ηfz}.

Choose z' in X such that {ξgz'}=Sz= {z}, then

$${}_{\delta}F_{z,Tz'}(\phi(u)) = {}_{\delta}F_{Sz,Tz'}(\phi(u))$$

$$\geq min\{F_{\eta fz,\xi gz'}(u), {}_{\delta}F_{\xi gz',Tz'}(u), {}_{\delta}F_{\xi gz',Sz}(\beta u),$$
$$[F_{\eta fz,\xi gz'} \oplus F_{\xi gz',Tz'}](2-\beta)(u), [F_{Sz,\eta fz}$$
$$\oplus F_{z',\xi gz'}](2-\beta)(u),$$
$$2.\ F_{\eta fz',\xi gz'}(u).\left[\frac{1+F_{z',\xi gz'}(u)}{1+F_{Sz,\xi gz'}(u)}\right]\}$$
$$= {}_{\delta}F_{z,Tz'}(u) \qquad (2.59)$$

Lemma 2.6, we get Tz' = {z}.

Since T and ξg are compatible and {ξgz'} = Tz' , we get Tξgz' = ξgTz' , that is, Tz = {ξgz}. Now, for u > 0,

$${}_{\delta}F_{Sz,Tz}(\phi(u))$$
$$\geq min\{F_{\eta fz,\xi gz}(u), {}_{\delta}F_{\xi gz,Tz}(u), {}_{\delta}F_{\xi gz,Sz}(\beta u),$$
$$[F_{\eta fz,\xi gz} \oplus F_{\xi gz,Tz}](2-\beta)(u), [F_{Sz,\eta fz}$$
$$\oplus F_{Tz,\xi gz}](2-\beta)(u),$$
$$2.\ F_{\eta fz,\xi gz}(u).\left[\frac{1+F_{Tz,\xi gz}(u)}{1+F_{Sz,\xi gz}(u)}\right]\}$$
$$= F_{\eta fz,\xi gz}(u) = {}_{\delta}F_{Sz,Tz}(u). \qquad (2.60)$$

So we have Sz = Tz = {*ηfz*} = {ξgz} = {z}.

Step 4. For u > 0 with β = 1 in the condition (v), we get

$${}_{\delta}F_{Sfz,Tx_{2n+1}}(\phi(u))$$
$$\geq min\{F_{\eta ffz,\xi gx_{2n+1}}(u), {}_{\delta}F_{\xi gx_{2n+1},Tx_{2n+1}}(u),$$
$${}_{\delta}F_{\xi gx_{2n+1},Sfz}(\beta u),$$
$$[F_{\eta ffz,\xi gx_{2n+1}} \oplus F_{\xi gx_{2n+1},Tx_{2n+1}}](2-\beta)(u),$$
$$[F_{Sfz,\eta ffz} \oplus F_{Tx_{2n+1},\xi gx_{2n+1}}](2-\beta)(u),$$
$$2.\ F_{Sfz,\xi gx_{2n+1}}(u).\left[\frac{1+F_{Tx_{2n+1},\xi gx_{2n+1}}(u)}{1+F_{Sfz,\xi gx_{2n+1}}(u)}\right]\}$$
$$(2.61)$$

By the condition (ii), *ηf* = *fη*, *Sf* = *f S*, so we have *ηf*(*f* z)=*f*(η*fz*)=*f* z and S(*fz*)={ *f* (Sz)}= { *fz*}. Taking $\lim_{n\to\infty}$ inf, by Lemma 2.8,

$$F_{fz,z}(\phi(u))$$
$$\geq min\{F_{fz,z}(u), F_{z,z}(u), F_{z,z}(\beta u),$$
$$[F_{fz,z} \oplus F_{z,z}](2-\beta)(u), [F_{fz,fz}$$
$$\oplus F_{z,z}](2-\beta)(u),$$
$$2.\ F_{fz,z}(u).\left[\frac{1+F_{z,z}(u)}{1+F_{fz,z}(u)}\right]\} \qquad (2.62)$$

So we get f z = z.

Hence, by Steps 1 and 4, we have η f z = z and f z = z, which implies ηz = z. Therefore,
{z} = { f z} = {ηz} = Sz.

Step 5. For u > 0 with β=1 in condition (v), we get

$${}_{\delta}F_{Sx_{2n},Tgz}(\phi(u))$$
$$\geq min\{F_{\eta fx_{2n},\xi ggz}(u), {}_{\delta}F_{\xi ggz,Tgz}(u), {}_{\delta}F_{\xi ggz,Sx_{2n}}(\beta u),$$
$$[F_{\eta fx_{2n},\xi ggz} \oplus F_{\xi ggz,Tgz}](2-\beta)(u), [F_{Sx_{2n},\eta fx_{2n}}$$
$$\oplus F_{Tgz,\xi ggz}](2-\beta)(u),$$
$$2.\ F_{Sx_{2n},\xi ggz}(u).\left[\frac{1+F_{Tgz,\xi ggz}(u)}{1+F_{Sx_{2n},\xi ggz}(u)}\right]\} \qquad (2.63)$$

Since Tg = gT and ξg = gξ, we have Tgz = {gTz} = {gz} and ξg(gz) = g(ξgz) = gz. Taking

$\lim_{n\to\infty}$ inf, by Lemma 2.8, we get

$$F_{z,gz}(\phi(u)) \geq min\{F_{z,gz}(u), F_{gz},_{gz}(u), F_{gz,z}(\beta u), [F_{z,gz} \oplus F_{gz,gz}](2-\beta)(u), [F_{z,z} \oplus F_{gz,gz}](2-\beta)(u), 2.\, F_{z,gz}(u). \left[\frac{1+F_{gz,gz}(u)}{1+F_{z,\xi ggz}(u)}\right]\} \quad (2.64)$$

So we get gz = z.

Hence, by Steps 3 and 5, we have $\xi gz = z$ and gz = z, which implies $\xi z = z$.
So we have {z} = {gz} = {ξz} = Tz.

Therefore, we have

$$\{z\}=\{fz\}=\{gz\}=\{\eta z\}=\{\xi z\}=Sz=Tz. \quad (2.65)$$

Last, we want to prove the uniqueness. Let y be the another common fixed point of η, f , ξ, g, S, and T. Then for u > 0,

$$F_{z,y}\varphi((u)) = {}_{\delta}F_{Sz,Ty}\varphi((u)) \geq \min\{F_{\eta fz},_{\xi gy}(u), {}_{\delta}F_{\xi gy},_{Ty}(u), {}_{\delta}F_{\xi gy},_{Sz}(\beta u), [F_{\eta fz},_{\xi gy} \oplus F_{\xi gy, Ty}](2-\beta)(u), [F_{Sz},_{\eta fz} \oplus F_{Ty, \xi gy}](2-\beta)(u), 2.\, F_{Sz},_{\xi gy}(u). \left[\frac{1+F_{Ty},_{\xi gy}(u)}{1+F_{Sy},_{\xi gy}(u)}\right]\}$$

$$\geq \min\{F_{z},_{y}(u), F_{y},_{y}(u), F_{y},_{z}(\beta u), [F_{z},_{y} \oplus F_{y,y}](2-\beta)(u), [F_{z},_{z}\ F_{y,y}](2-\beta)(u), 2.\, F_{z},_{y}(u). \left[\frac{1+F_{y,y}(u)}{1+F_{y,y}(u)}\right]\} \quad (2.66)$$

This implies y = z. We complete the proof.

If we take f = g = I, the identity map on X in Theorem 2.9, then we immediately have the following corollary.

Corollary 2.10. Let (X, ℑ, min) be a complete Menger space. Let η,ξ : X → X be two single-valued functions, and let S,T : X → B(X) be two set-valued functions. If the following conditions are satisfied:

(i) S(X) ⊂ ξ(X), T(X) ⊂ η(X),
(ii) η or ξ is continuous,
(iii) (S,η) and (T,ξ) are compatible,
(iv) for u > 0,

$${}_{\delta}F_{Sx,Ty}\varphi(u) \geq \min\{F_{\eta x},_{\xi y}(u), {}_{\delta}F_{\xi y},_{Ty}(u), {}_{\delta}F_{\xi y},_{Sx}(\beta u), [F_{\eta x},_{\xi y} \oplus F_{\xi y, Ty}](2-\beta)(u), [F_{Sx},_{\eta x} \oplus F_{Ty, \xi y}](2-\beta)(u), 2.\, F_{Sx},_{\xi y}(u). \left[\frac{1+F_{Ty},_{\xi y}(u)}{1+F_{Sy},_{\xi y}(u)}\right]\} \quad (2.67)$$

for all x, y ∈ X, β ∈ (0,2), where φ ∈ Φ, then η, ξ, S, and T have a unique common fixed point z in X. By the same process of the proof of Theorem 2.9.

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