ISSN 2231-346X (Print)

JUSPS-A Vol. 29(9), 378-387 (2017). Periodicity-Monthly

ISSN 2319-8044 (Online)

Section A

JOURNAL OF ULTRA SCIENTIST OF PHYSICAL SCIENCES

An International Open Free Access Peer Reviewed Research Journal of Mathematics

website:- www.ultrascientist.org

Estd. 1989

Generalized Salagean-Type Harmonic Univalent Functions with Negative Coefficients

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http://dx.doi.org/10.22147/jusps-A/290904

Acceptance Date 1st August, 2017, Online Publication Date 2nd September, 2017

Abstract

The purpose of this paper is to derive certain interesting properties involving coefficient inequalities, distortion bounds, extreme points, convex combination and radii of convexity for a new class of generalized Salagean-Type harmonic univalent functions in the open unit disc. Relevant connections of our results with various known results are briefly indicated.

Keywords and Phrases: Salagean operator, harmonic univalent functions.

2000 Mathematics Subject Classification: 30C45, 31A05.

1 Introduction

A continuous complex valued function $f(z) = u(x, y) + iv(x, y), z = x + iy$ defined in a simply connected domain $D \subset C$ is said to be harmonic in D if both u and v are real-valued harmonic functions in D. In a simply connected domain, we can write $f(z) = h(z) + \overline{g}(z)$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. An analytic univalent function is a sense preserving (or orientation preserving) map if it preserves angles between curves. Clunie and Sheil- Small[2] observed that a necessary and sufficient condition for f to be locally univalent and sense preserving in D is that

$$|h'(z)| > |g'(z)| \text{ in } D.$$

Let S_H denote the family of functions $f = h + \overline{g}$ that are harmonic univalent and sense preserving in the open unit disc $U = \{z : |z| < 1\}$ with

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \; g(z) = \sum_{k=1}^{\infty} b_k z^k, |b_1| < 1. \tag{1.1}$$

For $g = 0$, the harmonic function $f = h + \overline{g}$ reduces to analytic function h.

Jahangiri *et al.*[5] and Dixit and Porwal[3] have studied Salagean-type harmonic univalent functions. A generalized Salagean operator was considered by Al-Oboudi[11] in following way.

$$D^0 f(z) = f(z)$$

$$D' f(z) = (1-\lambda) f(z) + \lambda z f'(z) = D_\lambda f(z), \lambda \geq 0$$

$$D^n f(z) = D_\lambda (D^{n-1} f(z)).$$

It we put $\lambda = 1$, we have Salagean operator[7].

Thus, for $f = h + \overline{g}$ given by (1.1), we define the modified generalized Salagean operator of f as

$$D^m f(z) = D^m h(z) + (-1)^m D^m g(z), \quad (m \in N_0 = N \cup \{0\}) \tag{1.2}$$

where $$D^m h(z) = z + \sum_{k=2}^{\infty} [1 + (k-1)\lambda]^m a_k z^k,$$

$$D^m g(z) = z + \sum_{k=1}^{\infty} [1 + (k-1)\lambda]^m b_k z^k.$$

Now, for $0 \leq \alpha < 1, 0 \leq \mu < 1, m \in N, n \in N_0, m > n$ and $z \in U$, we define and investigate in this paper a family $S_H^\lambda(m,n;\alpha;\mu)$ of harmonic univalent functions $f = h + \overline{g}$ where h, g are given by (1.1) and

$$\operatorname{Re}\left\{ \frac{D^m f(z)}{\mu D^m f(z) + (1-\mu) D^n f(z)} \right\} > \alpha. \tag{1.3}$$

Where $D^m f$ is denoted by (1.2).

Further, let the subclass $\overline{S}_H^\lambda(m,n;\alpha;\mu)$ consist of harmonic univalent functions $f_m = h + \overline{g}_m$ in $S_H^\lambda(m,n;\alpha;\mu)$ so that h and g_m are of the form

$$h(z) = z - \sum_{k=2}^{\infty} a_k z^k, \qquad g_m(z) = (-1)^{m-1} \sum_{k=1}^{\infty} b_k z^k, \qquad a_k, b_k \geq 0 \tag{1.4}$$

If we put $\lambda = 1$, then it reduces to the class $S_H(m, n; a; m)$ studied by Dixit and Porwal[3] which is a generalization and an improvement of papers[1,4,5,6,8,9,10].

In the present paper, we obtain many results including coefficient estimates, extreme points, distortion theorems, convex combination and radii of convexity for the classes $S_H^{\lambda}(m, n; \alpha; \mu)$ and $\overline{S}_H^{\lambda}(m, n; \alpha; \mu)$ of harmonic univalent functions.

2 Main Results

In this section, first we will prove a sufficient condition for functions in $S_H^{\lambda}(m, n; \alpha; \mu)$.

Theorem 2.1 Let $f = h + \overline{g}$ be given by (1.1).

Further more

$$\sum_{k=1}^{\infty} \frac{\{1+(k-1)\lambda\}^m (1-\alpha\mu) - \alpha(1-\mu)\{1+(k-1)\lambda\}^n}{1-\alpha} |a_k|$$

$$+ \frac{\{1+(k-1)\lambda\}^m (1-\alpha\mu) - (-1)^{m-n}\alpha(1-\mu)\{1+(k-1)\lambda\}^n}{1-\alpha} |b_k| \le 2. \tag{2.1}$$

Where $a_1 = 1, m \in N, n \in N_0, m > n, 0 \le a < 1, 0 \le \mu < 1 \lambda \ge 1$, then f is sense preserving harmonic univalent in U and $f \in S_H^{\lambda}(m, n; \alpha; \mu)$.

Proof. If $z_1 \ne z_2$ then

$$\left|\frac{f(z_1) - f(z_2)}{h(z_1) - h(z_2)}\right| \ge 1 - \left|\frac{g(z_1) - g(z_2)}{h(z_1) - h(z_2)}\right|$$

$$= 1 - \left| \frac{\sum_{k=1}^{\infty} b_k \left(z_1^k - z_2^k\right)}{(z_1 - z_2) + \sum_{k=2}^{\infty} a_k \left(z_1^k - z_2^k\right)} \right|$$

$$> 1 - \frac{\sum_{k=1}^{\infty} k|b_k|}{1 - \sum_{k=2}^{\infty} k|a_k|}$$

$$\ge 1 - \frac{\sum_{k=1}^{\infty} \frac{\{1+(k-1)\lambda\}(1-\alpha\mu) - (-1)^{m-n}\alpha(1-\mu)}{1-\alpha}|b_k|}{1 - \sum_{k=2}^{\infty} \frac{\{1+(k-1)\lambda\}(1-\alpha\mu) - \alpha(1-\mu)}{1-\alpha}|a_k|}$$

$$\geq 1-\frac{\left[\sum_{k=1}^{\infty}\frac{\{1+\lambda(k-1)\}^{n}\{(1+\lambda(k-1))(1-\alpha\mu)-(-1)^{m-n}\alpha(1-\mu)\}}{1-\alpha}|b_k|\right]}{1-\sum_{k=2}^{\infty}\frac{\{1+\lambda(k-1)\}^{n}\{(1+\lambda(k-1))(1-\alpha\mu)-\alpha(1-\mu)\}}{1-\alpha}|a_k|}$$

$$\geq 1-\frac{\left[\sum_{k=1}^{\infty}\frac{\{1+\lambda(k-1)\}^{n+1}(1-\alpha\mu)-(-1)^{m-n}\alpha(1-\mu)\{1+\lambda(k-1)\}^{n}}{1-\alpha}|b_k|\right]}{1-\sum_{k=2}^{\infty}\frac{\{1+\lambda(k-1)\}^{n+1}(1-\alpha\mu)-\alpha(1-\mu)\{1+\lambda(k-1)\}^{n}}{1-\alpha}|a_k|}$$

$$\geq 1-\frac{\left[\sum_{k=1}^{\infty}\frac{\{1+\lambda(k-1)\}^{m}(1-\alpha\mu)-(-1)^{m-n}\alpha(1-\mu)\{1+\lambda(k-1)\}^{n}}{1-\alpha}|b_k|\right]}{1-\sum_{k=2}^{\infty}\frac{\{1+\lambda(k-1)\}^{m}(1-\alpha\mu)-\alpha(1-\mu)\{1+\lambda(k-1)\}^{n}}{1-\alpha}|a_k|}$$

Sincem > n,

≥ 0 [Using(2.1)],

Which proves the univalance.

Also, we have

$$|h'(z)| > 1-\sum_{k=2}^{\infty}k|a_k||z|^{k-1} > 1-\sum_{k=2}^{\infty}k|a_k|$$

$$\geq 1-\sum_{k=2}^{\infty}\frac{\{1+\lambda(k-1)\}^{m}(1-\alpha\mu)-\alpha(1-\mu)\{1+\lambda(k-1)\}^{n}}{1-\alpha}|a_k|$$

$$\geq \sum_{k=1}^{\infty}\frac{\{1+\lambda(k-1)\}^{m}(1-\alpha\mu)-(-1)^{m-n}\alpha(1-\mu)\{1+\lambda(k-1)\}^{n}}{1-\alpha}|b_k|$$

$$\geq \sum_{k=1}^{\infty}k|b_k||z|^{k-1} \geq |g'(z)|.$$ Hencefisse nsepreservinginU.

Using the fact $\operatorname{Re}(w) > \alpha$ iff $|1-\alpha+w|>|1+\alpha-w|$, it suffices to show that

$$\left|(1-\alpha)\{\mu D^m f(z)+(1-\mu)D^n f(z)\}+D^m f(z)\right|$$
$$-\left|(1+\alpha)\{\mu D^m f(z)+(1-\mu)D^n f(z)\}-D^m f(z)\right| > 0. \quad (2.2)$$

Substituting for $D^m(z)$ and $D^n f(z)$ in L.H.S. of (2.2) we have

$$|(2-\alpha)z+\sum_{k=2}^{\infty}[(1-\alpha)\{\mu(1+\lambda(k-1))^m+(1-\mu)(1+\lambda(k-1))^n\}+(1+\lambda(k-1))^m]a_k z^k$$

$$+(-1)^n\sum_{k=1}^{\infty}[(-1)^{m-n}(1-\alpha)\mu\{1+\lambda(k-1)\}^m+(1-\alpha)(1-\mu)\{1+\lambda(k-1)\}^n$$

$$+(-1)^{m-n}(1+\lambda(k-1))^m]\times b_k z^{-k}|$$

$$-|\alpha z+\sum_{k=2}^{\infty}[(1+\alpha)[\mu\{1+\lambda(k-1)\}^m+(1-\mu)\{1+\lambda(k-1)\}^n]-\{1+\lambda(k-1)\}^m]a_k z^k$$

$$+(-1)^n\sum_{k=1}^{\infty}[(-1)^{m-n}(1+\alpha)\mu\{1+\lambda(k-1)\}^m+(1+\alpha)(1-\mu)\{1+\lambda(k-1)\}^n-(-1)^{m-n}$$

$$\{1+\lambda(k-1)\}^m]b_k z^{-k}|$$

$$\geq 2(1-\alpha)|z|-\sum_{k=2}^{\infty}2[\{1+\lambda(k-1)\}^m(1-\alpha\mu)-\alpha(1-\mu)\{1+\lambda(k-1)\}^n]\times a_k|z|^k$$

$$-\sum_{k=1}^{\infty}(-1)^{m-n}[(1-\alpha)\mu.\{1+\lambda(k-1)\}^m+\{1+\lambda(k-1)\}^m]$$

$$+(1-\alpha)(1-\mu)\{1+\lambda(k-1)\}^n\|b|_k|z|^k$$

$$-\sum_{k=1}^{\infty}(-1)^{m-n}[(1+\alpha)\mu\{1+\lambda(k-1)\}^m-\{1+\lambda(k-1)\}^m]+(1+\alpha)(1-\mu)$$

$$\times\{1+\lambda(k-1)^n\}\|b|_k|z|^k$$

$$=2(1-\alpha)|z|-2\sum_{k=2}^{\infty}[\{1+\lambda(k-1)\}^m(1-\alpha\mu)-\alpha(1-\mu)\{1+\lambda(k-1)\}^n\times|a_k\||z|^k]$$

$$-2\sum_{k=2}^{\infty}[\{1+\lambda(k-1)\}^m(1-\alpha\mu)+\alpha(1-\mu)\{1+\lambda(k-1)\}^n]\times|b_k\||z|^k$$

$$[\text{If}(\text{m}-\text{n})\text{ is odd}]$$

$$=2(1-\alpha)|z|-2\sum_{k=2}^{\infty}[\{1+\lambda(k-1)\}^m(1-\alpha\mu)-\alpha(1-\mu)\{1+\lambda(k-1)\}^n]\times|a_k\||z|^k$$

$$-2\sum_{k=2}^{\infty}[\{1+\lambda(k-1)\}^m(1-\alpha\mu)-\alpha(1-\mu)\{1+\lambda(k-1)\}^n]\times|b_k\||z|^k$$

$$[\text{If}(\text{m}-\text{n})\text{ is even}]$$

$$= 2(1-\alpha)|z|[1-\sum_{k=2}^{\infty}\frac{\{1+\lambda(k-1)\}^m(1-\alpha\mu)-\alpha(1-\mu)\{1+\lambda(k-1)\}^n}{(1-\alpha)}|a_k||z|^{k-1}.$$

$$-\sum_{k=1}^{\infty}.\frac{\{1+\lambda(k-1)\}^m(1-\alpha\mu)-(-1)^{m-n}\alpha(1-\mu)\{1+\lambda(k-1)\}^n}{(1-\alpha)}|b_k||z|^{k-1}]$$

$$\geq 2(1-\alpha)[1-\sum_{k=2}^{\infty}\frac{\{1+\lambda(k-1)\}^m(1-\alpha\mu)-\alpha(1-\mu)\{1+\lambda(k-1)\}^n|a_k|}{(1-\alpha)}.$$

$$-\sum_{k=1}^{\infty}.\frac{\{1+\lambda(k-1)\}^m(1-\alpha\mu)-(-1)^{m-n}\alpha(1-\mu)\{1+\lambda(k-1)\}^n}{(1-\alpha)}|b_k|].$$

The last expression in non-negative by (2.1) and so that proof of Theorem (2.1) is established. The harmonic univalent functions

$$f(z)=z+\sum_{k=2}^{\infty}\frac{(1-\alpha)}{\{1+\lambda(k-1)\}^m(1-\alpha\mu)-\alpha(1-\mu)\{1+\lambda(k-1)\}^n}x_k z^k$$

$$+\sum_{k=1}^{\infty}\frac{(1-\alpha)}{\{1+\lambda(k-1)\}^m(1-\alpha\mu)-(-1)^{m-n}\alpha(1-\mu)\{1+\lambda(k-1)\}^n}\overline{y_k z^k} \tag{2.3}$$

where $m\in N, n\in N_0, m>n$ and $\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k|=1$
shows that coefficient bound given by (2.1) is sharp. □

Following theorem shows that the assertion (2.1) is also necessary for functions of the type $f_m=h+\overline{g}_m$, Where h and g_m are of the form (1.4).

Theorem 2.2 Let $f_m=h+\overline{g}_m$ be given by (1.4). Then $f_m\in\overline{S}_H^{\lambda}(m,n;\alpha;\mu)$ if and only if

$$\sum_{k=1}^{\infty}\Big[\{1+\lambda(k-1)\}^m(1-\alpha\mu)-\alpha(1-\mu)\{1+\lambda(k-1)\}^n a_k$$

$$+\Big[\{1+\lambda(k-1)\}^m(1-\alpha\mu)-(-1)^{m-n}\alpha(1-\mu)\{1+\lambda(k-1)\}^n b_k\leq 2(1-\alpha) \tag{2.4}$$

Proof. Since $\overline{S}_H^{\lambda}(m,n;\alpha;\mu)\subset S_H^{\lambda}(m,n;\alpha;\mu)$, we only need to prove the "only if" part of the theorem. For functions of the form (1.4), we see that the assertion

$$Re\left\{\frac{D^m f_m(z)}{\mu D^m f_m(z)+(1-\mu)D^n f_m(z)}\right\}>\alpha \quad \text{isequivalent to}$$

$$Re\left[\frac{\begin{array}{c}(1-\alpha)z-\sum_{k=2}^{\infty}[\{1+\lambda(k-1)\}^m(1-\alpha\mu)-\alpha(1-\mu)\{1+\lambda(k-1)\}^n]a_kz^k\\ +(-1)^{2m-1}\sum_{k=1}^{\infty}[\{1+\lambda(k-1)\}^m(1-\alpha\mu)-(-1)^{m-n}\\ \alpha(1-\mu)\{1+\lambda(k-1)\}^n]b_kz^{-k}\end{array}}{\begin{array}{c}z-\sum_{k=2}^{\infty}[\mu\{1+\lambda(k-1)\}^m+(1-\mu)\{1+\lambda(k-1)\}^n]a_kz^k\\ +(-1)^{2m-1}\sum_{k=1}^{\infty}[\mu\{1+\lambda(k-1)\}^m+(-1)^{m-n}\\ (1-\mu)\{1+\lambda(k-1)\}^n]b_kz^{-k}\end{array}}\right]\geq 0 \qquad (2.5)$$

This required assertion (2.5) must hold for all values of z in U. Upon choosing the values of z on the positive real axis where $0\leq|z|=r<1$, we must have

$$\frac{\begin{array}{l}(1-\alpha)-\sum_{k=2}^{\infty}\{1+\lambda(k-1)\}^m(1-\alpha\mu)-\alpha(1-\mu)\{1+\lambda(k-1)\}^n\Big]a_kr^{k-1}\\ -\sum_{k=1}^{\infty}\Big[\{1+\lambda(k-1)\}^m(1-\alpha\mu)-(-1)^{m-n}\alpha(1-\mu)\{1+\lambda(k-1)\}^n\Big]b_kr^{k-1}\end{array}}{\begin{array}{l}1-\sum_{k=2}^{\infty}\Big[\mu\{1+\lambda(k-1)\}^m+(1-\mu)\{1+\lambda(k-1)\}^n\Big]a_kr^{k-1}\\ -\sum_{k=1}^{\infty}\Big[\mu\{1+\lambda(k-1)\}^m+(-1)^{m-n}(1-\mu)\{1+\lambda(k-1)\}^n\Big]b_kr^{k-1}\end{array}}\geq 0. \qquad (2.6)$$

If the condition (2.4) does not hold, then numerator in (2.6) is negative for r sufficiently close to 1. Hence, there exists $z_0=r$ in $(0,1)$ for which the quotient in (2.6) is negative. This contradicts the required condition for $f_m\in\overline{S}_H^{\lambda}(m,n;\alpha;\mu)$ and so the proof is complete. □

Next, we determine the extreme points of closed convex hull of $\overline{S}_H^{\lambda}(m,n;\alpha;\mu)$ denoted by clco $\overline{S}_H^{\lambda}(m,n;\alpha;\mu)$.

Theorem 2.3 Let f_m be given by (1.4). Then $f_m\in\overline{S}_H^{\lambda}(m,n;\alpha;\mu)$, iff

$$f_m(z)=\sum_{k=1}^{\infty}\{x_kh_k(z)+y_kg_{mk}(z)\},\ \text{where}\ h_1(z)=z$$

$$h_k(z)=z-\frac{(1-\alpha)}{\{1+\lambda(k-1)\}^m(1-\alpha\mu)-\alpha(1-\mu)\{1+\lambda(k-1)\}^n}z^k\ (k=2,3,4,....)$$

$$g_{mk}(z) = z + (-1)^{m-1} \frac{(1-\alpha)z^{-k}}{\{1+\lambda(k-1)\}^m (1-\alpha\mu) - (-1)^{m-n} \alpha (1-\mu) \{1+\lambda(k-1)\}^n}, (k = 1,2,3,4,...)$$

$x_k \geq 0, y_k \geq 0, \sum_{k=1}^{\infty}(x_k + y_k) = 1$. In particular, the extreme points of $\overline{S}_H^{\lambda}(m,n;\alpha;\mu)$ are $\{h_k\}$ and $\{g_{mk}\}$.

Proof. The proof of Theorem 2.3 is much akin to that of Theorem 2.3 of [3], and therefore we omit the details involved. □

Now we determine the distortion bounds for functions in $\overline{S}_H^{\lambda}(m,n;\alpha;\mu)$ which yield a covering result for this class.

Theorem 2.4 Let $f_m \in \overline{S}_H^{\lambda}(m,n;\alpha;\mu)$. Then for $|z| = r < 1$, we have

$$|f_m(z)| \leq (1+b_1)r + \frac{1}{(1+\lambda)} n \left[\frac{(1-\alpha)}{(1+\lambda)^{m-n}(1-\alpha\mu) - 2(1-\mu)} - \frac{(1-\alpha\mu)-(-1)^{m-n}\alpha(1-\mu)}{(1+\lambda)^{m-n}(1-\alpha\mu)-\alpha(1-\mu)} b_1 \right] r^2, |z| = r < 1, \text{ and}$$

$$|f_m(z)| \geq (1+b_1)r - \frac{1}{(1+\lambda)^n} \left[\frac{(1-\alpha)}{(1+\lambda)^{m-n}(1-\alpha\mu) - \alpha(1-\mu)} - \frac{(1-\alpha\mu)-(-1)^{m-n}\alpha(1-\mu)}{(1+\lambda)^{m-n}(1-\alpha\mu)-\alpha(1-\mu)} b_1 \right] r^2, |z| = r < 1$$

Proof. The details involved are simply routine work and may be omitted. □

This result enables us to find covering results from the left hand inequality in Theorem 2.4.

Theorem 2.5 If $f_m \in \overline{S}_H^{\lambda}(m,n;\alpha;\mu)$, then

$$\left\{ w : |w| < \frac{(1+\lambda)^m - 1 - \alpha[(1+\lambda)^m \mu + (1+\lambda)^n (1-\mu) - 1]}{(1+\lambda)^m (1-\alpha\mu) - \alpha(1-\mu)(1+\lambda)^n} \right.$$

$$\left. - \frac{(1+\lambda)^m - 1 - \alpha[(1+\lambda)^m \mu + (1+\lambda)^n (1-\mu) - \lambda - (-1)^{m-n}(1-\mu)]}{(1+\lambda)^m (1-\alpha\mu) - \alpha(1-\mu)(1+\lambda)^n} b_1 \right\} \subset f(U)$$

Remark 1. The covering results in the above theorem coincides with that obtained in[3] for $\lambda = 1$.

We can show that $\overline{S}_H^{\lambda}(m,n;\alpha;\mu)$ is closed under convex combination by adopting standard techniques.

Theorem 2.6. The class $\overline{S}_H^{\lambda}(m,n;\alpha;\mu)$ is closed under convex combination.

Theorem 2.7. If $f_m \in \overline{S}_H^{\lambda}(m,n;\alpha;\mu)$, then f_m is convex in the disc

$$|z| \leq \min_k \left\{ \frac{(1-\alpha)(1-b_1)}{k[(1-\alpha) - \{(1-\alpha\mu) - (-1)^{m-n}\alpha(1-\mu)\} b_1} \right\}^{\frac{1}{k-1}} \quad (k = 2,3,4,.......)$$

Proof. Let $f_m \in \overline{S}_H^{\lambda}(m,n;\alpha;\mu)$ and let $r(0<r<1)$ fixed. Then $r^{-1}f(rz) \in \overline{S}_H^{\lambda}(m,n;\alpha;\mu)$ and we have

$$\sum_{k=2}^{\infty} k^2(a_k+b_k)r^{k-1}$$

$$=\sum_{k=2}^{\infty} k(a_k+b_k)(kr^{k-1})$$

$$\le \sum_{k=2}^{\infty}\left[\frac{\{1+\lambda(k-1)\}^m(1-\alpha\mu)-\alpha(1-\mu)\{1+\lambda(k-1)\}^n}{(1-\alpha)}a_k\right.$$

$$\left.+\frac{\{1+\lambda(k-1)\}^m(1-\alpha\mu)-(-1)^{m-n}\alpha(1-\mu)\{1+\lambda(k-1)\}^n}{(1-\alpha)}b_k\right]kr^{k-1}$$

$$\le\left[1-\left\{\frac{(1-\alpha\mu)-(-1)^{m-n}\alpha(1-\mu)}{(1-\alpha)}\right\}b_1\right]kr^{k-1}\le 1-b_1$$

provided $kr^{k-1}\le \dfrac{1-b_1}{1-\dfrac{(1-\alpha\mu)-(-1)^{m-n}\alpha(1-\mu)}{1-\alpha}b_1}$,

which is true if

$$r\le \min_k\left\{\frac{(1-\alpha)(1-b_1)}{k\left[(1-\alpha)-\{(1-\alpha\mu)-(-1)^{m-n}\alpha(1-\mu)\}b_1\right]}\right\}\quad (k=2,3,4,\ldots)$$ □

For our next theorem we need to define the convolution of two harmonic functions. For harmonic functions of the form

$$f_m(z)=z-\sum_{k=2}^{\infty}a_k z^k+(-1)^{m-1}\sum_{k=1}^{\infty}b_k\bar{z}^{-k}$$

and $$F_m(z)=z-\sum_{k=2}^{\infty}A_k z^k+(-1)^{m-1}\sum_{k=1}^{\infty}B_k\bar{z}^{-k}$$

we define the convolution of two harmonic functions f_m and F_m as

$$(f_m*F_m)(z)=f_m(z)*F_m(z)=z-\sum_{k=2}^{\infty}a_kA_kz^k+(-1)^{m-1}\sum_{k=1}^{\infty}b_kB_k\bar{z}^{-k}.$$

By using this definition, we show that the class $\overline{S}_H^{\lambda}(m,n;\alpha;\mu)$ is closed under convolution.

Theorem 2.8. For $0 \leq \beta \leq \alpha < 1$, let $f_m \in \overline{S}_H^{\lambda}(m,n;\alpha;\mu)$ and $F_m \in \overline{S}_H^{\lambda}(m,n;\beta;\mu)$. Then $f_m * F_m \in \overline{S}_H^{\lambda}(m,n;\alpha;\mu) \subseteq \overline{S}_H^{\lambda}(m,n;\beta;\mu)$.

Proof. Argument is similar to that of Theorem 2.6 of [3], and therefore details involved may be omitted.

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