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Generalized binary closed sets in binary topological spaces

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Abstract

Recently the authors introduced the concept of binary topology between two sets and investigate its basic properties where a binary topology from X to Y is a binary structure satisfying certain axioms that are analogous to the axioms of topology. In this paper we introduce and study generalized binary closed sets and generalized binary open sets that are analogous to the generalized closed sets and generalized open sets in point set topology.

Key words: Binary topology, binary open, binary closed, binary closure, binary subspace, generalized binary closed sets and generalized binary open sets.

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1. Introduction

Recently the authors[2] introduced the concept of binary topology and discussed some of its basic properties. The purpose of this paper is to introduce generalized binary closed sets and generalized binary open sets in binary topological spaces and characterize their basic properties. Section 2 deals with basic concepts. Generalized binary closed sets in binary topological spaces are discussed in section 3. Section 4 deals with generalized binary open sets in binary topological spaces. Throughout the paper, $\wp(X)$ denotes the power set of X.

2. Preliminaries :

Let X and Y be any two nonempty sets. A binary topology[2] from X to Y is a binary structure $\mathcal{M} \subseteq \wp(X) \times \wp(Y)$ that satisfies the axioms namely (i) $(\varnothing,\varnothing)$ and $(X,Y) \in \mathcal{M}$, (ii) $(A_1 \cap A_2, B_1 \cap B_2) \in \mathcal{M}$ whenever $(A_1, B_1) \in \mathcal{M}$ and $(A_2, B_2) \in \mathcal{M}$, and (iii) If $\{(A_\alpha, B_\alpha) : \alpha \in \Delta\}$ is a family of members of $\mathcal{M}$, then $\left(\bigcup_{\alpha\in\Delta} A_\alpha, \bigcup_{\alpha\in\Delta} B_\alpha\right) \in \mathcal{M}$. If $\mathcal{M}$ is a binary topology from X to Y then the triplet $(X, Y, \mathcal{M})$ is called a binary topological space and the members of $\mathcal{M}$ are called the binary open subsets of the binary topological space $(X,Y,\mathcal{M})$. The elements of $X\times Y$ are called the binary points of the binary topological space $(X,Y,\mathcal{M})$. If $Y=X$ then $\mathcal{M}$ is called a binary topology on X in which case we write $(X,\mathcal{M})$ as a binary topological space. The examples of binary topological spaces are given[1,3,5].

Definition[2] *2.1.* Let X and Y be any two nonempty sets and let (A,B) and (C,D) $\in \wp(X)\times\wp(Y)$. We say that $(A,B)\subseteq(C,D)$ if $A\subseteq C$ and $B\subseteq D$.

Proposition 2.2.[2] Let $(X,Y,\mathcal{M})$ be a binary topological space and $(A,B)\subseteq(X,Y)$.

Let $(A,B)^{1*}=\cap\{A_\alpha : (A_\alpha,B_\alpha)$ is binary closed and $(A,B)\subseteq(A_\alpha,B_\alpha)\}$ and $(A,B)^{2*}=\cap\{B:(A_\alpha,B_\alpha)$ is binary closed and $(A,B)\subseteq(A_\alpha,B_\alpha)\}$. Then $((A,B)^{1*},(A,B)^{2*})$ is binary closed and $(A,B)\subseteq((A,B)^{1*},(A,B)^{2*})$.

Definition 2.3.[2] The ordered pair $((A,B)^{1*},(A,B)^{2*})$ is called the binary closure of (A,B), denoted by b-*cl*(A, B) in the binary space $(X,Y,\mathcal{M})$ where $(A,B)\subseteq(X,Y)$.

Definition 2.4.[4] Let $(X,Y,\mathcal{M})$ be a binary topological space. Let $(A,B)\subseteq(X,Y)$. Define $\mathcal{M}_{(A,B)}=\{(A\cap U, B\cap V):(U,V)\in\mathcal{M}\}$. Then $\mathcal{M}_{(A,B)}$ is a binary topology from A to B. The binary topological space $(A,B,\mathcal{M}_{(A,B)})$ is called a binary sub-space of $(X,Y,\mathcal{M})$.

Definition 2.5. Let X and Y be any two nonempty sets and let (A,B) and (C,D) $\in\wp(X)\times\wp(Y)$. We say that $(A,B)\not\subset(C,D)$ if one of the following holds :

(i) $A\subseteq C$ and $B\not\subset D$ (ii) $A\not\subset C$ and $B\subseteq D$ (iii) $A\not\subset C$ and $B\not\subset D$.

3. *Generalized binary closed sets :*

Definition 3. 1. Let $(X,Y,\mathcal{M})$ be a binary topological space. Let $(A,B)\in\wp(X)\times\wp(Y)$. Then (A,B) is called generalized binary closed if b-*cl*$(A,B)\subseteq(U,V)$ whenever $(A,B)\subseteq(U,V)$ and (U,V) is binary open in $(X,Y,\mathcal{M})$.

Example 3. 2. Let X = {a, b,c}. Let $\mathcal{M}$= {(∅,∅),({a},{c}),({c},{b}),({b},{a}),({a,b},{a, c}),({b, c},{a,b}),({a,c},{b,c}), (X, X)}. Then $\mathcal{M}$ is a binary topology on X.

Now, the binary closed sets are (∅,∅), ({b,c},{a,b}),({a,b},{a,c}),({a,c},{b,c}), ({a},{c}),({c},{b}),({b},{a}),(X,X). Consider ({c},{a,b}). Clearly ({c},{a.b}) is generalized binary closed, since ({c},{a,b})⊆({b,c},{a,b}) where ({b,c},{a,b}) is binary open and b-*cl*({c},{a,b})= ({b,c},{a,b}) ⊆({b,c},{a,b}). Also ({c},{a,b}) is not binary closed. Consider ({a},{c}). Clearly ({a},{c}) is generalized binary closed, since ({a},{c})⊆({a,b},{a,c}) where ({a,b},{a,c}) is binary open and b-*cl*({a},{c})= ({a},{c}) ⊆({a,b},{a,c}). Also ({a},{c}) is binary closed.

Example 3.3. Consider X = {a, b}, Y= {1, 2,3}. $\wp(X)$ = {∅, {a},{b}, X}. $\wp(Y)$={∅, {1},{2},{3},{1,2},{1,3},{2,3}, Y}.

Now, $\mathcal{M}$={(∅,∅),(∅,{1}),({a},{1}), ({a},{1,2}),({b},∅),({b},{1}), ({b},{3}), ({b},{1,3}), ({X, {1}), (X,{1,2}), (X,{1,3}), (X,Y)}. Then $\mathcal{M}$ is a binary topology from X to Y. Consider ({a},{1,3}). Clearly ({a},{1,3}) is not generalized binary closed, since ({a}, {1,3}) ⊆(X,{1,3}) where (X,{1,3}) is binary open and b-*cl*({a},{1,3})=({a},Y)⊄(X,{1,3}).

Also ({c},{1,3}) is not binary closed.

Consider ({a},{2}). Clearly ({a},{2}) is generalized binary closed, since ({a},{2})⊆ ({a}, {1,2}) where ({a},{1,2}) is binary open and b-*cl*({a},{2})=({a},{2}) ⊆({a}{1,2}). Also ({a},{2}) is binary closed. We easily verify all the binary closed sets are generalized binary closed.
The following Proposition is easily proved.

Proposition 3. 4. Binary closed sets in a binary topological space are generalized binary closed.
The converse of the Proposition 3.4 is need not true, for in Example 3. 2 the binary set ({c},{a,b}) is generalized binary closed but not binary closed.

Proposition 3. 5. Let (A,B) be a generalized binary closed set in a binary topological space (X,Y, $\mathcal{M}$) .Suppose (C,D)⊆ $((A,B)^{1*}\backslash A, (A,B)^{2*}\backslash B)$ where (C,D) is binary closed in (X,Y,$\mathcal{M}$). Then (C,D)=(∅,∅).

Proof. Let (C,D) ⊆$((A,B)^{1*}\backslash A,(A,B)^{2*}\backslash B)$ where (C,D) is binary closed in (X,Y, $\mathcal{M}$). Therefore, (C,D) ⊆ $((A,B)^{1*},(A,B)^{2*})$. Now, (A,B)⊆ (X\C,Y\D) and since (A,B) is generalized binary closed, we have b-*cl*(A,B)⊆ (X\C,Y\D). That is, $((A,B)^{1*}, (A,B)^{2*})$)⊆ (X\C,Y\D) or (C,D) ⊆$(X\backslash(A,B)^{1*}, Y\backslash(A,B)^{2*})$. This gives (C,D) ⊆ $((A,B)^{1*} \cap(X\backslash(A,B)^{1*}),$ $(A,B)^{2*}\cap(Y\backslash(A,B)^{2*}))$ = (∅,∅).
This implies (C,D)= (∅, ∅).

Proposition 3.6. Let (X,Y, $\mathcal{M}$) be a binary topological space. Let (A,B)∈ (X)× ℘(Y). Suppose $((A,B)^{1*}\backslash A, (A,B)^{2*}\backslash B)$ contains the only binary closed set (∅,∅), then (A,B) is generalized binary closed.

Proof. Let (A,B) ⊆(U,V) where (U,V) is binary open in (X,Y, $\mathcal{M}$). Suppose $(A,B)^{1*}$ is not a subset of U and $(A,B)^{2*}$ is not a subset of V. Hence $((A,B)^{1*}, (A,B)^{2*})$ is not a subset of (U,V). This implies $((A,B)^{1*} ,(A,B)^{2*})$ ⊆ (X\U,Y\V). Therefore, $((A,B)^{1*} \cap(X\backslash U),$ $(A,B)^{2*} \cap(Y\backslash V))$ is a binary closed set and $(A,B)^{1*} \cap(X\backslash U) \neq \emptyset$, $(A,B)^{2*} \cap(Y\backslash V)\neq \emptyset$.

Also $((A,B)^{1*} \cap(X\backslash U), (A,B)^{2*} \cap(Y\backslash V))$ ⊆ $((A,B)^{1*}\backslash A, (A,B)^{2*}\backslash B)$. This is a contradiction. Hence b-*cl*(A,B)⊆ (U,V).

Proposition 3.7. A generalized binary closed set (A,B) is binary closed if and only if $((A,B)^{1*}\backslash A, (A,B)^{2*}\backslash B)$ is binary closed.

Proof. Let (A,B) be a generalized binary closed set. Assume that (A,B) is binary closed .
We shall show that $((A,B)^{1*}\backslash A,(A,B)^{2*}\backslash B)$ is binary closed.
Since (A,B) is binary closed, we have b-*cl*(A,B)= (A,B). Therefore, $((A,B)^{1*} , (A,B)^{2*}$)=(A,B). This implies $((A,B)^{1*}\backslash A, (A,B)^{2*}\backslash B)$= (∅, ∅) which is binary closed .
Conversely assume that $((A,B)^{1*}\backslash A, (A,B)^{2*}\backslash B)$ is binary closed. But (A,B) is generalized binary closed and $((A,B)^{1*}\backslash A, (A,B)^{2*}\backslash B)$ is

a binary closed subset of itself.
By Proposition 3.5 $((A,B)^{1*} \backslash A, (A,B)^{2*} \backslash B) = (\varnothing,\varnothing)$. Therefore, $((A,B)^{1*}, (A,B)^{2*}) = (A,B)$. That is, b-$cl(A,B) = (A,B)$. By Proposition 3.6 in[2], we have (A,B) is binary closed.

Proposition 3.8. Let $(X,Y,\mathcal{M})$ be a binary topological space. Let $(A,B) \subseteq (X,Y)$ and

$(A,B,\mathcal{M}_{(A,B)})$ be a binary subspace of $(X,Y,\mathcal{M})$. Suppose $(C,D)\subseteq(A,B)\subseteq(X,Y)$ and (C,D) is generalized binary closed in $(X,Y,\mathcal{M})$. Then (C,D) is generalized binary closed relative to (A,B).

Proof. Let $(C,D)\subseteq(A\cap U,B\cap V)$ where (U,V) is binary open in $(X,Y,\mathcal{M})$. Then $(C,D)\subseteq(U,V)$.
Since (C,D) is generalized binary closed in $(X,Y,\mathcal{M})$, we have b-$cl(C,D)\subseteq (U,V)$. Therefore, $((C,D)^{1*},(C,D)^{2*})\subseteq(U,V)$. This implies $((C,D)^{1*} \cap A, (C,D)^{2*}\cap B) \subseteq (A\cap U, B\cap V)$. Hence, (C,D) is generalized binary closed relative to (A,B).

Theorem 3.9. In a binary topological space $(X,Y,\mathcal{M})$, $\mathcal{M} = \mathcal{M}^{\prime}$ (the set of all binary closed sets) if and only if every binary set $(A,B)\subseteq(X,Y)$ is generalized binary closed.

Proof. Suppose $\mathcal{M} = \mathcal{M}^{\prime}$. Let $(A,B)\subseteq (U,V)\in\mathcal{M}$.

Then (by Proposition[2] 3.7) b-$cl(A,B) \subseteq$ b-$cl(U,V)=(U,V)$. This implies (A,B) is generalized binary closed. Conversely assume that every binary set $(A,B) \subseteq(X,Y)$ is generalized binary closed. Let $(U,V)\in\mathcal{M}$.

Since $(U,V)\subseteq(U,V)$ and (U,V) is generalized binary closed,
we have b-$cl(U,V)\subseteq(U,V)$. By definition, we have $(U,V)\subseteq$b-$cl(U,V)$.

This implies b-$cl(U,V)=(U,V)$. Hence, $(U,V)\in\mathcal{M}^{\prime}$. Thus, $\mathcal{M}\subseteq\mathcal{M}^{\prime}$. Let $(U^{\prime}, V^{\prime}) \in \mathcal{M}^{\prime}$. Then $(X\backslash U^{\prime},Y\backslash V^{\prime})\in\mathcal{M}\subseteq\mathcal{M}^{\prime}$. Therefore $(U^{\prime},V^{\prime})\in \mathcal{M}^{\prime}$. This implies $\mathcal{M}^{\prime}\subseteq \mathcal{M}$.

Proposition 3.10. Let $(X,Y,\mathcal{M})$ be a binary topological space. Suppose (C,D) $(A,B)\subseteq (X,Y)$ and (C,D) is generalized binary closed relative to (A,B) and (A,B) is generalized binary closed subset of (X,Y). Then (C,D) is generalized binary closed relative to (X,Y).

Proof. Let $(C,D)\subseteq(U,V)$ where (U,V) is binary open in $(X,Y,\mathcal{M})$. Then $(C,D)\subseteq (A\cap U, B\cap V)$. Since (C,D) is generalized binary closed relative to (A,B), we have b-$cl_{(A,B)}(C,D)\subseteq(A\cap U, B\cap V)$. Hence, $((C,D)^{1*}, (C,D)^{2*})_{(A,B)}\subseteq(A\cap U, B\cap V)$. This implies $(A\cap(C,D)^{1*}, B\cap(C,D)^{2*})\subseteq(A\cap U,B\cap V)$ and $(A,B)\subseteq(U\cup (X\backslash (C,D)^{1*}), V\cup (Y\backslash(C,D)^{2*}))$. Since (A,B) is generalized binary closed in (X,Y), we have b-$cl(A,B) \subseteq (U\cup (X\backslash(C,D)^{1*}), V\cup (Y\backslash(C,D)^{2*}))$. Also, since $(C,D)\subseteq(A,B)$, we have b-$cl(C,D) \subseteq$ b-$cl(A,B) \subseteq(U\cup (X\backslash (C,D)^{1*}),V\cup(Y\backslash(C,D)^{2*}))$. Therefore, b-$cl(C,D) \subseteq(U,V)$. This shows that (C,D) is generalized binary closed relative to $(X,Y,\mathcal{M})$.

Proposition 3.11. Let (A,B) be generalized binary closed and suppose that (C,D) is binary closed. If $(A\cap C, B\cap D)$

$\subseteq(U,V)$ where (U,V) is binary open, then $((A,B)^{1*}\cap C, (A,B)^{2*}\cap D)\subseteq(U,V)$.

Proof. Let $(A\cap C, B\cap D)\subseteq(U,V)$ where (U,V) is binary open. Therefore, $(A, B)\subseteq(U,V)$ and (C, D) (U,V). Since (A,B) is generalized binary closed, we have b-cl$(A, B)\subseteq(U,V)$.
That is, $((A,B)^{1*}, (A,B)^{2*})\subseteq(U,V)$. Therefore, $((A,B)^{1*}\cap C, (A,B)^{2*}\cap D)\subseteq(U,V)$.

Proposition 3.12: If (A,B) is generalized binary closed and $(A,B)\subseteq(C,D))\subseteq$ b-cl(A,B), then (C,D) is generalized binary closed.

Proof. Now $(A,B)\subseteq(C,D)\Rightarrow$b-cl$(A,B)\subseteq$b-cl$(C,D)$ $\Rightarrow((A,B)^{1*}, (A,B)^{2*})\subseteq((C,D)^{1*}, (C,D)^{2*})$ $\Rightarrow((C,D)^{1*}\backslash C,(C,D)^{2*}\backslash D)\subseteq((A,B)^{1*}\backslash A, (A,B)^{2*}\backslash B)$

By Proposition 3.5, $((C,D)^{1*}\backslash C, (C,D)^{2*}\backslash D)) = (\varnothing,\varnothing)$. This implies $((C,D)^{1*},(C,D)^{2*}) =(C,D)$. Hence, b-cl$(C,D)=(C,D)$. Thus (C,D) is binary closed. By Proposition 3.4, (C,D) is generalized binary closed.

4 Generalized binary open sets :

Definition 4. 1. Let $(X,Y, \mathcal{M})$ be a binary topological space. Let $(A,B)\in\wp(X)\times\wp(Y)$. Then (A,B) is called generalized binary open if $(X\backslash A,Y\backslash B)$ is generalized binary closed in $(X,Y, \mathcal{M})$.

Example 4.2. Consider $X=\{a, b\}$, $Y=\{1, 2,3\}$. Now, $\mathcal{M}=\{(\varnothing,\varnothing),(\varnothing,\{1\}),(\{a\},\{1\}), (\{a\},\{1,2\}),(\{b\},\varnothing),(\{b\},\{1\}), (\{b\},\{3\}), (\{b\},\{1,3\}), (\{X,\{1\}),(X,\{1,2\}),(X,\{1,3\}), (X,Y)\}$. Then $\mathcal{M}$ is a binary topology from X to Y. The binary set $(\{b\},\{1,3\})$ is generalized binary open, since $(X\backslash\{b\},Y\backslash\{1,3\})= (\{a\},\{2\})\in\wp(X)\times\wp(Y)$ is generalized binary closed, for $(\{a\},\{2\})\subseteq(\{a\},\{1,2\})$ where $(\{a\},\{1,2\})$ is binary open and b-$cl(\{a\},\{2\})=(\{a\},\{2\})\subseteq(\{a\}\{1,2\})$

Proposition 4.3. Every binary open set in a binary topological space is generalized binary open.

Proof. Let $(X,Y,\mathcal{M})$ be a binary topological space. Let (A,B) be binary open in $(X,Y,\mathcal{M})$. Then $(X\backslash A,Y\backslash B)$ is binary closed in $(X,Y,\mathcal{M})$. By Proposition3.4, we have $(X\backslash A,Y\backslash B)$ is generalized binary closed in $(X,Y,\mathcal{M})$. Therefore, (A,B) is generalized binary open in $(X,Y,\mathcal{M})$.

The converse of the Proposition 4.3 is need not true, for Example consider $X = \{a, b\}$ and $Y=\{1, 2,3\}$. Now, $\mathcal{M}=\{(\varnothing,\varnothing), (\varnothing,\{1\}), (\{a\},\{1\}), (X,Y)\}$. Clearly $\mathcal{M}$ is a binary topology from X to Y. Consider $(\{b\},\{1\})$. Clearly $(\{b\},\{1\})$ is generalized binary closed, since $(\{b\},\{1\})\subseteq(X,Y)$ where (X,Y) is binary open and b-$cl(\{b\},\{1\})=(X,Y)\subseteq(X,Y)$.
Hence $(\{a\},\{2,3\})$ is generalized binary open.
But $(\{a\},\{2,3\})$ is not binary open in $(X,Y,\mathcal{M})$.

Conclusion

Generalized closed sets and generalized open sets in topological spaces are extended to binary topological spaces. Some elementary properties of generalized binary closed sets and generalized binary open sets are discussed.

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