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On **gs- Homeomorphism In Topological Spaces

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Abstract

In this paper we introduce a new class of closed maps namely **gs-closed maps which settled in between the class of *gs-closed maps[18] and the class of gs-closed maps[8]. We also introduce and study new class of homeomorphisms called **gs-homeomorphisms and **gs*-homeomorphisms. Further we show that the set of all **gs*-homeomorphisms form a group under the operation composition of maps.

Key words and phrases: **gs-closed maps; **gs-homeomorphisms; **gs*-homeomorphisms.

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1. Introduction

The notion homeomorphism plays an important role in topology. A homeomorphism between two topological spaces X and Y is a bijective mapping $f : X \to Y$ when both f and f^{-1} are continuous. Malghan[22] and Devi *et al.*[8] introduced the concept of generalized closed maps and semi generalized closed maps respectively in topological spaces. Manoj *et al.*[21] introduced the concept of **gs-closed sets in topological spaces. In this paper we first introduce a new class of closed maps namely **gs-closed maps and then introduce and study **gs-homeomorphisms and **gs*-homeomorphisms in a topological space. We also prove that the set of all **gs*-homeomorphisms forms a group under the operation composition of functions.

2. Preliminaries :

Throughout this paper (X, τ), (Y, σ) and (Z, η) represent topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of space (X, τ) the cl(A), int(A) and A^c denote the closure of A, the interior of A and the complement of A in X respectively.

We recall the following definitions:

Definition 2.01 : A subset A of a topological space (X, τ) is called semi-open[1] (resp. semi-closed, semi pre-open[3]) if A $\subseteq$ cl(int(A)) (resp. int(cl(A)) $\subseteq$ A, A $\subseteq$ cl(int(cl(A))).

Definition 2.02: A subset A of a topological space (X, τ) is called g-closed[2] (resp. sg-closed[4], gs-closed[5], *g-closed[13], ψ-closed[11], *gs-closed[17], $\hat{\hat{g}}$-closed[12], **g-closed[15], gsp-closed[9], **gs-closed[21]) set if cl(A) $\subseteq$ U (resp. scl(A) $\subseteq$ $\subseteq$ U, cl(A) $\subseteq$ U, scl(A) $\subseteq$ U, scl(A) $\subseteq$ U, cl(A) $\subseteq$ U, cl(A) $\subseteq$ U, spcl(A) $\subseteq$ U, scl(A) $\subseteq$ U) whenever A $\subseteq$ U and U is open (resp. semi open, open, $\hat{g}$-open, sg-open, $\hat{g}$-open, sg-open, $\hat{\hat{g}}$-open, $\hat{\hat{g}}$-open) set in (X, τ).

Definition 2.03 : A map f : (X, τ) $\to$ (Y, σ) is called semi-closed[20] (resp. g-closed[22], sg-closed[8], gs-closed[8], ψ-closed[19], *g-closed[13], *gs-closed[18], $\hat{\hat{g}}$-closed[12], **g-closed[15], gsp-closed[9]) map if the image of each closed set in (X, τ) is semi closed (resp. g-closed, sg-closed, gs-closed, ψ-closed, *g-closed, *gs-closed, $\hat{\hat{g}}$-closed, **g-closed, gsp-closed) in (Y, σ).

Definition 2.04: A function f : (X, τ) $\to$ (Y, σ) is called g-continuous[6] (resp. semi-continuous[8], sg-continuous[7], gs-continuous[10], ψ-continuous[11], *g-continuous[13], *gs-continuous[17], $\hat{\hat{g}}$-continuous[12], **g-continuous[15], gsp-continuous[9]) if the inverse image of every σ-closed set in Y is g-closed (resp. semi closed, sg-closed, gs-closed, ψ-closed, *g-closed, *gs-closed, $\hat{\hat{g}}$-closed, **g-closed, gsp-closed) set in X.

Definition 2.05 : A bijection f : (X, τ) $\to$ (Y, σ) is called (i) semi-homeomorphism(B)[16] if f is continuous and semi-open (ii) sg-homeomorphism[(10)] if f is both sg-continuous and sg-open (iii) gs-homeomorphism[10] if f is both gs-continuous and gs-open (iv) $\hat{\hat{g}}$-homeomorphism[14] if f is both $\hat{\hat{g}}$-continuous and $\hat{\hat{g}}$-open (v) *g-homeomorphism[13] if f is both *g-continuous and *g-open (vi) *gs-homeomorphism[18] if f is both *gs-continuous and *gs-open (vii) ψ-homeomorphism[19] if f is both ψ-continuous and ψ-open (viii) **g-homeomorphism[15] if f is both **g-continuous and **g-open.

Definition 2.06: Let (X, τ) be a topological space and A $\subseteq$ X. Then **gs-closure of A (briefly **gs-cl(A)[12]) is defined as the intersection of all **gs-closed sets containing A.

*3. **gs-Closed Maps :*

In this section we introduce the notions of **gs-closed maps, **gs-open maps, **gs*-closed maps and **gs*-open maps in topological spaces and obtained certain characterizations of these maps.

Definition 3.01: A map f : (X, τ) $\to$ (Y, σ) is called **gs-closed (resp. **gs-open) map if f(A) is **gs-closed (resp. **gs-open) set in (Y, σ) for every closed (open) set A of (X, τ).

Definition 3.02: Let (X, τ) be a topological space and A $\subseteq$ X. We define the **gs-interior of A (briefly **gs-int(A)) to be the union of all **gs-open sets contained in A.

Theorem 3.01: Every closed map, semi-closed map, *g-closed map, ψ-closed map, *gs-closed map, **g-closed map is **gs-closed map.

Next examples show that the converse of the above theorem is not true in general.

Example 3.01: Let X = Y = {a, b, c}, τ = {ϕ, {b}, {c}, {b, c}, X} and σ = {ϕ, {b}, {a, c}, Y}. Define f : (X, τ) $\to$ (Y,

σ) by identity mapping then f is not closed map, semi-closed map, *g-closed map, ψ-closed map and *gs-closed maph, however f is **gs-closed map.

Example 3.02: Let X = Y = {a, b, c}, τ = {ϕ, {a}, {b}, {a, b}, {a, c}, X} and σ = {ϕ, {a}, {a, b}, Y}. Define f : (X, τ) $\rightarrow$ (Y, σ) by identity mapping then f is not a **g-closed map, however f is **gs-closed map.

Theorem 3.02: Every **gs-closed map is gs-closed map.

Therefore the class of **gs-closed maps properly contains the class of closed maps, the class of semi-closed maps, the class of *g-closed maps, the class of ψ-closed maps, the class of *gs-closed maps and the class of **g-closed maps and properly contained in class of gs-closed maps.

Theorem 3.03: A map f : (X, τ) $\rightarrow$ (Y, σ) is **gs-closed iff **gs-cl(f(A)) $\subseteq$ f(cl(A)) for every subset A of (X, τ).

Theorem 3.04: A map f : (X, τ) $\rightarrow$ (Y, σ) is **gs-closed iff for each subset A of (Y, σ) and for each open set U containing $f^{-1}(A)$ there exists a **gs-open set V of (Y, σ) such that A $\subseteq$ V and $f^{-1}(V) \subseteq U$.

Proof: Let f is **gs-closed mapping. Let A $\subseteq$ Y and U be an open subset of (X, τ) such that $f^{-1}(A) \subseteq$ U then $V = (f(U^C))^C$ is a **gs-open set containing A such that $f^{-1}(V) \subseteq U$.

Conversely let A be a closed set in X then f $n^{-1}((f(A))^C) \subseteq A^C$ and A^C is open in X. By assumption, there exists a **gs-open set V of (Y, σ) s.t. $(f(A))^C \subseteq V$ and $f^{-1}(V) \subseteq A^C$ so $A \subseteq (f^{-1}(V))^C$. Hence $V^C \subseteq f(A) \subseteq f((f^{1}(V))^C) \subseteq V^C$ i.e. $f(A) = V^C$, since V^C is **gs-closed so f(A) is **gs-closed i.e. f is **gs-closed.

Theorem 3.05: If f : (X, τ) $\rightarrow$ (Y, σ) be a closed map and g : (Y, σ) $\rightarrow$ (Z, η) be a **gs-closed map then their composition gof : (X, τ) $\rightarrow$ (Z, η) is **gs-closed map.

Remark 3.01: The following example shows that the composition of two **gs-closed maps need not be **gs-closed map.

Example 3.03: Let X = {a, b, c}, τ = {ϕ, {a}, {b}, {a, b}, {a, c}, X}, σ = {ϕ, {a}, {a, b}, X} and η = {ϕ, {a, b}, X}. Define f : (X, τ) $\rightarrow$ (X, σ) by identity mapping and g : (X, σ) $\rightarrow$ (X, η) by identity mapping then f and g both are **gs-closed maps but their composition gof : (X, τ) $\rightarrow$ (X, η) is not a **gs-closed map.

Theorem 3.06: If f : (X, τ) $\rightarrow$ (Y, σ) and g : (Y, σ) $\rightarrow$ (Z, η) be two mappings such that their composition gof : (X, τ) $\rightarrow$ (Z, η) be a **gs-closed map then the following are true

(i) If f is continuous and surjective, then g is **gs-closed map.

(ii) If g is **gs-irresolute and injective, then f is **gs-closed map.

Theorem 3.07: If f : (X, τ) $\rightarrow$ (Y, σ) be a **gs-closed map and g : (Y, σ) $\rightarrow$ (Z, η) be a closed map, then their composition gof : (X, τ) $\rightarrow$ (Z, η) need not be a **gs-closed map.

The following example supports the above theorem.

Example 3.04: Let X = Y = Z = {a, b, c}, τ = {ϕ, {a}, {b, c}, X}, σ = {ϕ, {a, c}, Y} and η = {ϕ, {b}, {c}, {b, c}, {a, c}, Z}. Define f : (X, τ) $\rightarrow$ (Y, σ) by f(a) = b, f(b) = b, f(c) = c and g : (Y, σ) $\rightarrow$ (Z, η) by identity mapping. Then f is **gs-closed map and g is closed map but their composition gof : (X, τ) $\rightarrow$ (Z, η) is not a **gs-closed map.

Theorem 3.08: For any bijective f : (X, τ) $\rightarrow$ (Y, σ) the following statements are equivalent.

(i) f^{-1} : (Y, σ) $\rightarrow$ (X, τ) is **gs-continuous.

(ii) f is **gs-open map and

(iii) f is **gs-closed map.

Theorem 3.09: For any A $\subseteq$ X, int(A) $\subseteq$ **gs-int(A) $\subseteq$ A.

Proof: Since every open set is **gs-open so proof is obvious.

Theorem 3.10: Let f : (X, τ) $\rightarrow$ (Y, σ) be a **gs-closed map then for a subset A of (X, τ), f(int(A)) $\subseteq$ **gs-int(f(A)).

Theorem 3.11: A function f : (X, τ) $\rightarrow$ (Y, σ) is **gs-open if and only if for any subset B of (Y, σ) and for any closed set C containing $f^{-1}(B)$, there exists a **gs-closed set A of (Y, σ) containing B such that $f^{-1}(A) \subseteq C$.

Theorem 3.12: A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is **gs-open if and only if f^{-1}(**gs-cl(B)) $\subseteq$ cl(f^{-1}(B) for every subset B of (Y, σ).

Definition 3.03: A map $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be a **gs*-closed (resp. **gs*-open) if the image f(A) is **gs-closed (resp. **gs-open) set in (Y, σ) for every **gs-closed (resp. **gs-open) set A in (X, τ).

Theorem 3.13: Every **gs*-closed map is **gs-closed map.

The converse is not true in general as it can be seen from the following example.

Example 3.05: Let X = Y = {a, b, c}, τ = {ϕ, {a}, {b, c}, X} and σ = {ϕ, {b}, {c}, {b, c}, {a, c}, Y}. Define $f : (X, \tau) \rightarrow (Y, \sigma)$ by f(a) = b, f(b) = a and f(c) = c then f is **gs-closed map but not **gs*-closed map.

Theorem 3.14: For any bijection $f : (X, \tau) \rightarrow (Y, \sigma)$ the following are equivalent

(i) $f^{-1} : (Y, s) \rightarrow (X, \tau)$ is **gs-irresolute,

(ii) f is a **gs*-open map and

(iii) f is a **gs*-closed map.

Theorem 3.15: A map $f : (X, \tau) \rightarrow (Y, \sigma)$ is **gs*-closed iff **gs-cl(f(A)) $\subseteq$ f(**gs-cl(A)) for every subset A of (X, τ).

*4. **gs-Homeomorphisms*

In this section we introduce the following definitions.

Definition 4.01: A bijection $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be **gs-homeomorphisms if f is **gs-continuous and **gs-open maps.

Theorem 4.01: Every semi-homeomorphism (B) and so homeomorphism is **gs-homeomorphism.

The converse of the above theorem is not true in general as it can be seen from the following example.

Example 4.01: Let X = Y = {a, b, c}, τ = {ϕ, {b}, X} and σ = {ϕ, {c}, {b, c}, Y}. Define $f : (X, \tau) \rightarrow (Y, \sigma)$ f(a) = a, f(b) = c, f(c) = b. Then f is **gs-homeomorphism but not homeomorphism and semi homeomorphism (B).

Theorem 4.02: Every *g-homeomorphism, ψ-homeomorphism, *gs-homeomorphism, $\hat{\hat{g}}$-homeomorphism and **g-homeomorphism is **gs-homeomorphism.

The converse of the above theorem is not true in general as it can be seen from the following example.

Example 4.02: Let X = Y = {a, b, c}, τ = {ϕ, {a, c}, X} and σ = {ϕ, {a}, Y}. Define $f : (X, \tau) \rightarrow (Y, \sigma)$ by identity mapping then f is **gs-homeomorphism but not *g-homeomorphism, ψ-homeomorphism, *gs-homeomorphism, $\hat{\hat{g}}$-homeomorphism.

Example 4.03: Let X = Y = {a, b, c}, τ = {ϕ, {a}, {c}, {a, c}, X} and σ = {ϕ, {a}, {c}, {b, c}, {a, c}, Y}. Define $f : (X, \tau) \rightarrow (Y, \sigma)$ by identity mapping then f is **gs-homeomorphism but not **g-homeomorphism.

Definition 4.02: A bijection $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be **gs-homeomorphism if both f and f^{-1} are **gs-irresolute.

We denote the family of all **gs*-homeomorphism of a topological space (X, τ) onto itself by **gs*-h(X, τ).

Theorem 4.04: Every **gs*-homeomorphism is **gs-homeomorphism.

The converse of the above theorem is not true in general as it can be seen from the following example.

Example 4.04: Let X = Y = {a, b, c}, τ = {ϕ, {a}, X} and σ = {ϕ, {a}, {a, b}, Y}. Define $f : (X, \tau) \rightarrow (Y, \sigma)$ by identity mapping then f is **gs-homeomorphism but not **gs*-homeomorphism.

Therefore the class of **gs-homeomorphism properly contains the class of homeomorphism, the class of semi-homeomorphism (B), the class of *g-homeomorphism, the class of ψ-homeomorphism, the class of *gs-homeomorphism, the class of $\hat{\hat{g}}$-homeomorphism and the class of **g-homeomorphism. Also this new class is properly contained in the class of gs-homeomorphism.

Theorem 4.05: If f : $(X, \tau) \rightarrow (Y, \sigma)$ and g : $(Y, \sigma) \rightarrow (Z, \eta)$ are **gs*-homeomorphism, then their composition gof : $(X, \tau) \rightarrow (Z, \eta)$ is also **gs*-homeomorphism.

Theorem 4.06: If f : $(X, \tau) \rightarrow (Y, \sigma)$ be a bijective and **gs-continuous map then following statements are equivalent

(i) f is **gs-open map.

(ii) f is **gs-homeomorphism.

(iii) f is **gs-closed map.

Theorem 4.07: The set **gs*-h(X, τ) is a group under the composition of maps.

Proof: Define a binary operation, □ : **gs*-h(X, τ) □ **gs*-h(X, τ) by f □ g = g $_o$ f for all f and g □ **gs*-h(X, τ) and o is the usual operation of composition of maps then by theorem (4.05), g □ f $\in$ **gs*-h(X, τ). Again composition of maps is associative and the identity map I : $(X, \tau) \rightarrow (X, \tau)$ belonging to **gs*-h(X, τ) serves as the identity element of **gs*-h(X, τ). If f $\in$ **gs*-h(X, τ), then $f^{-1} \in$ **gs*-h(X, τ) such that f □ $f^{-1} = f^{-1}$ □ f = I and so inverse exists for each element of **gs*-h(X, τ). Thus (**gs*-h(X, τ), □) is a group under the operation of composition of maps.

Theorem 4.08: If f : $(X, \tau) \rightarrow (Y, \sigma)$ be a **gs*-homeomorphism, then f induces an isomorphism from the group **gs*-h(X, τ) onto the group **gs*-h(Y, σ).

Proof: Define ψ_f : **gs*-h$(X, \tau) \rightarrow$ **gs*-h(Y, σ) by $\psi_f(h) = fohof^{-1}$ for every h $\in$ **gs*-h(X, τ). Then ψ_f is a bijection. Further, for all $h_1, h_2 \in$ **gs*-h(X, τ), $\psi_f(h_1oh_2) = fo(h_1oh_2)of^{-1} = (foh_1of^{-1})\ o\ (foh_2of^{-1}) = \psi_f(h_1)$ o $\psi_f(h_2)$ so ψ_f is a homeomorphism and so it is an isomorphism induced by f.

Theorem 4.09: **gs*-homeomorphism is an equivalence relation in the collection of all topological spaces.

Proof: Reflexivity and symmetry are immediate and transitivity follows from theorem (4.07).

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