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On mixed trilateral generating functions of extended Jacobi polynomials

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Abstract

In this note we have obtained some novel result on mixed trilateral relations involving extended Jacobi polynomials by group theoretic method which inturn yields the corresponding results involving Hermite, Laguerre and Jacobi polynomials.

Key words: Generating functions, Jacobi polynomials, Hermite polynomials, Laguerre polynomials.

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1. Introduction

The extended Jacobi polynomial is defined by [1]:

$$F_n(\alpha,\beta;x) = \frac{(-1)^n}{n!}\left(\frac{\lambda}{b-a}\right)^n (x-a)^{-\alpha}(b-x)^{-\beta} \times D^n\left[(x-a)^{n+\alpha}(b-x)^{n+\beta}\right], \quad (1.1)$$

where $D = \dfrac{d}{dx}$ and λ is a number such that $\dfrac{\lambda}{b-a} > 0$.

In this paper we have encountered somenovel result on mixed trilateral relations involving $F_n(\alpha,\ \beta+n;\ x)$-a modified form of $F_n(\alpha,\ \beta;\ x)$ and derived an unified presentation of a classof mixed trilateral generating relations for certain special functions.

The main result of our investigation is stated in the form of the following theorem.

Theorem 1: If there exists a bilateral generating relation of the form:

$$G(x,u,w) = \sum_{n=0}^{\infty} a_n F_n(\alpha,\ \beta+n;\ x)\, g_n(u)\, w^n, \quad (1.2)$$

where $g_n(u)$ is an arbitrary polynomial of degree n, then

$$\left\{1-\frac{\lambda}{b-a}(x-b)w\right\}^{-1-\alpha-\beta}(1+\lambda w)^{\beta}$$

$$\times G\left(\frac{x-\frac{a\lambda}{b-a}(x-b)w}{1-\frac{\lambda}{b-a}(x-b)w},u,\frac{wz(1+\lambda w)}{\left\{1-\frac{\lambda}{b-a}(x-b)w\right\}^{2}}\right)$$

$$=\sum_{n=0}^{\infty}w^{n}\,\sigma_n(x,u,z), \tag{1.3}$$

where

$$\sigma_n(x,u,z)=\sum_{p=0}^{n}a_p\binom{n}{p}F_n(\alpha,\,\beta-n+2p;\,x)\,g_p(u)\,z^{p}.$$

The importance of the above theorem lies in the fact that whenever one knows a generating relation of the form (1.2) then the corresponding mixed trilateral generating relation can at once be written down from (1.3). So one can get a large number of mixed trilateral generating relations by attributing different suitable values to a_n in (1.2).

1. Proof of the theorem

Wenowconsider the following partial differential operator R [2]:

$$R=\frac{\lambda}{b-a}\left[\frac{(x-a)(x-b)z}{y^2}\frac{\partial}{\partial x}+\frac{(x-a)z}{y}\frac{\partial}{\partial y}+(2x-a-b)\frac{z^2}{y^2}\frac{\partial}{\partial z}\right.$$
$$\left.+(1+\alpha)(x-b)\frac{z}{y^2}\right] \tag{2.1}$$

such that

$$R\left(F_n(\alpha,\beta+n;x)y^{\beta}z^{n}\right)=(n+1)F_{n+1}(\alpha,\beta+n-1;x)y^{\beta-2}z^{n+1}. \tag{2.2}$$

The extended form of the group generated by R is given by

$$e^{wR}f(x,\,y,\,z)=\left\{1-(x-b)\frac{\lambda w}{b-a}\,\frac{z}{y^2}\right\}^{-1-\alpha}$$

$$\times f\left(\frac{x-(x-b)\frac{\lambda aw}{b-a}\,\frac{z}{y^2}}{1-(x-b)\frac{\lambda w}{b-a}\,\frac{z}{y^2}},\ \frac{y\left(1+\lambda w\frac{z}{y^2}\right)}{1-(x-b)\frac{\lambda w}{b-a}\,\frac{z}{y^2}},\ \frac{z\left(1+\lambda w\,\frac{z}{y^2}\right)}{\left\{1-(x-b)\frac{\lambda w}{b-a}\,\frac{z}{y^2}\right\}^{2}}\right). \tag{2.3}$$

Let us assume the following bilateral generating relation of the form:

$$G(x,u,w)=\sum_{n=0}^{\infty} a_n F_n(\alpha,\beta+n;x)\, g_n(u)\, w^n. \tag{2.4}$$

Replacing w by wz and multiplying both sides of (2.4) by y^β and finally operating e^{wR} on both sides, we get

$$e^{wR}\left(y^\beta\, G(x,u,wz)\right)=e^{wR}\left(\sum_{n=0}^{\infty} a_n\left(F_n(\alpha,\beta+n;x)y^\beta z^n\right) g_n(u)w^n\right). \tag{2.5}$$

Now the left member of (2.5), with the help of (2.3), reduces to

$$\left\{1-(x-b)\frac{\lambda\, w}{b-a}\,\frac{z}{y^2}\right\}^{-1-\alpha-\beta} y^\beta\left(1+\lambda w\frac{z}{y^2}\right)^\beta$$
$$\times G\left(\frac{x-(x-b)\dfrac{\lambda a w}{b-a}\,\dfrac{z}{y^2}}{1-(x-b)\dfrac{\lambda\, w}{b-a}\,\dfrac{z}{y^2}},u,\frac{wz\left(1+\lambda w\,\dfrac{z}{y^2}\right)}{\left\{1-(x-b)\dfrac{\lambda\, w}{b-a}\,\dfrac{z}{y^2}\right\}^2}\right). \tag{2.6}$$

The right member of (2.5), with the help of (2.2), becomes

$$=\sum_{n=0}^{\infty}\sum_{p=0}^{n} a_{n-p}(wz)^n\binom{n}{p}F_n(\alpha,\beta+n-2p;x)\, g_{n-p}(u)y^{\beta-2p}. \tag{2.7}$$

Now equating (2.6) and (2.7) and then substituting $\frac{z}{y^2}=1$, we get

$$\left\{1-\frac{\lambda}{b-a}(x-b)w\right\}^{-1-\alpha-\beta}(1+\lambda w)^\beta\, G\left(\frac{x-\dfrac{a\lambda}{b-a}(x-b)w}{1-\dfrac{\lambda}{b-a}(x-b)w},u,\frac{wz(1+\lambda w)}{\left\{1-\dfrac{\lambda}{b-a}(x-b)w\right\}^2}\right)$$

$$=\sum_{n=0}^{\infty} w^n\,\sigma_n(x,u,z),$$

where

$$\sigma_n(x,u,z)=\sum_{p=0}^{n} a_p\binom{n}{p}F_n(\alpha,\ \beta-n+2p;\ x)\ g_p(u)z^p.$$

This completes the proof of the theorem.

2. Special cases:

We now discuss some special cases of interest.

Special cases 1(On Laguerre polynomials):

Putting $a = 0, \lambda = 1$ and $b = \beta$ in Theorem 1and then simplifying and finally taking limit as $\beta \to \infty$, we get the following results on mixed trilateral generating functions involving Laguerre polynomials:

Theorem 2: If there exists a bilateral generating relation of the form:

$$G(x,u,w) = \sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(x)\, g_n(u)\, w^n$$

then

$$(1-w)^{-1-\alpha} \exp\left(\frac{-xw}{1-w}\right) G\left(\frac{x}{1-w}, u, \frac{wz}{1-w}\right) = \sum_{n=0}^{\infty} w^n L_n^{(\alpha)}(x) \sigma_n(u,z), \quad (1.3)$$

where

$$\sigma_n(u,z) = \sum_{p=0}^{n} a_p \binom{n}{p} g_p(u)\, z^p,$$

which is found derived[3,4].

Special cases 2(OnHermite polynomials):

Similarly, putting $\alpha = \beta, -a = b = \sqrt{\alpha}$ and $\lambda = \frac{2}{\sqrt{\alpha}}$ in Theorem 1, then simplifying and finally taking limit as $\alpha \to \infty$, we get the following result on mixed trilateral generating relations involving Hermite polynomials:

Theorem 3: If there exists a bilateral generating relation of the form:

$$G(x,u,w) = \sum_{n=0}^{\infty} a_n \frac{H_n(x)}{n!} g_n(u)\, w^n$$

then

$$\exp(2xw - w^2)\, G(x-w, u, wz) = \sum_{n=0}^{\infty} \frac{w^n}{n!} H_n(x) \sigma_n(u,z), \quad (1.3)$$

where

$$\sigma_n(u,z) = \sum_{p=0}^{n} a_p \binom{n}{p} g_p(u)\, z^p,$$

which is found derived[5,6].

Special cases 3 (On Jacobi polynomials):

Again, if we Put $-a = b = 1, \ \lambda = 1$ and interchanging α, β in Theorem 1, we get the following results on mixed trilateral generating relations involving Jacobi polynomials:

Theorem 4: If there exists a bilateral generating relation of the form:

$$G(x,u,w) = \sum_{n=0}^{\infty} a_n P_n^{(\alpha+n,\beta)}(x)\, g_n(u)\, w^n$$

then

$$\left\{1+\frac{w}{2}(1-x)\right\}^{-1-\alpha-\beta} (1+w)^{\alpha}\, G\left(\frac{x-\frac{w}{2}(1-x)}{1+\frac{w}{2}(1-x)}, u, \frac{wz(1+w)}{\left\{1+\frac{w}{2}(1-x)\right\}^2}\right) = \sum_{n=0}^{\infty} w^n \sigma_n(x,u,z),$$

where

$$\sigma_n(x,u,z) = \sum_{p=0}^{n} a_p \binom{n}{p} P_n^{(\alpha-n+2p,\beta)}(x)\, g_p(u) z^p,$$

which is found derived[7].

Finally, we would like to point it out

that we can get a resultfound derived[7] which is analogous to case 3 above, if we use the following symmetry relation:

$$P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x).$$

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