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Unique Common Fixed Point Theorem On Weak Commuting Mapping

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Abstract

The aim of this paper is to establish a new fixed point theorem on complete metric space for weak commuting mapping. Our results generalize several corresponding relations of weak commuting mapping in metric space.

Key words : Fixed point, weak commuting mapping, metric space.

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1. Introduction

In 1982 Sessa[8] defined weak commutativity in the theorem of Jungck[3] and its various generalizations by introducing the weak commutativity. Fisher[2] proved following theorem for two commuting mappings S and T.

Theorem 1.1: If S is a mapping and T is a continuous mapping of the complete metric space X into itself and satisfying the inequality:

$$d(ST,TSy) \le k[d(Tx,TSy)+d(Sy,STx)], \quad (1.1)$$

for all $x, y \in X$,where $0 \le k \le 1/2$, then S and T have a unique common fixed point.

Fisher further extended his theorem and proved a common fixed point of commuting mappings S and T.

Theorem 1.2. If S is a mapping and T is a continuous mapping of the complete metric space X into itself and satisfying the inequality:

$$d(ST,TSy) \le k[d(Tx,STx)+d(Sy,TSy)], \quad (1.2)$$

for all $x, y \in X$, where $0 \le k \le 1/2$, then S and T have a unique common fixed point.

Ranganathan[7] has further generalized

the result of Jungck[3] which gives criteria for the existence of a fixed point. Jungck[4] introduced again more generalized commutativity, the so called compatibility,which is more general than that of weak commutativity.After that Jungck[5] coined the term of compatible mappings in order to generalize the concept of weak commutativity. Weak commuting mapping received much attention in recent years[1,9,11].

2. Weak Commuting Mappings and Common Fixed Point

In this section Weak commuting mappings and unique common fixed point theorem in metric space is established. First we give the following definitions:

Definition 2.1. According to Sessa[8] two self maps S and T defined on metric space (X,d) are said to be weakly commuting maps if and only if $d(STx,TSx)\le d(Sx,Tx)$, for all $x\in X$

Definition 2.2. Two self mappings S and T of metric space (X,d) are said to be weak** commuted, if $S(X)\subset T(X)$ and for any $x\in X$,

$$\begin{aligned} d(S^2T^2x,T^2S^2x) &\le d(S^2Tx,TS^2x)\\ &\le d(ST^2x,T^2Sx)\le d(STx,TSx)\\ &\le d(S^2x,T^2x) \end{aligned}$$

Definition 2.3. A map $S:X\to X$, X being a metric space, is called an idempotent, if $S^2=S$.

Example 2.1. Let $X=[0,1]$ with Euclidean metric space and define S and T by $Sx=\dfrac{x}{x+5}, Tx=\dfrac{x}{5}$ for all $x\in X$, then $[0,9/10]\subset[0,1]$ where $Tx=[0,1]$ and $Sx=[0,9/10]$

$$\begin{aligned} d(S^2T^2x,T^2S^2x) &= \frac{x}{6x+625}-\frac{x}{150x+625}\\ &= \frac{144x^2}{(6x+625)(150x+625)}\\ &\le \frac{24x^2}{(6x+125)(30x+125)}\\ &= \frac{x}{6x+125}-\frac{x}{30x+125}\\ &= d(S^2Tx,TS^2x). \end{aligned}$$

Implies that $d(S^2T^2x,T^2S^2x)\le d(S^2Tx,TS^2x)$

$$\begin{aligned} d(S^2Tx,TS^2x) &= \frac{x}{6x+125}-\frac{x}{30x+125}\\ &= \frac{24x^2}{(6x+125)(30x+125)}\\ &\le \frac{24x^2}{(x+125)(25x+125)}\\ &= \frac{x}{x+125}-\frac{x}{25x+125}\\ &= d(ST^2x,T^2Sx) \end{aligned}$$

$d(S^2Tx,TS^2x)\le d(ST^2x,T^2Sx)$

$$d(ST^2x, T^2Sx) = \frac{x}{x+125} - \frac{x}{25x+125}$$

$$d(ST^2x, T^2Sx) = \frac{24x^2}{(x+125)(5x+125)}$$

$$\leq \frac{4x^2}{(x+25)(5x+25)}$$

$$= \frac{x}{x+25} - \frac{x}{5x+25}$$

$$= d(STx, TSx)$$

$$d(ST^2x, T^2Sx) \leq d(STx, TSx)$$

$$d(STx, TSx) = \frac{x}{x+25} - \frac{x}{5x+25}$$

$$= \frac{4x^2}{(x+25)(5x+25)}$$

$$\leq \frac{6x^2}{25(6x+25)}$$

$$= \frac{x}{25} - \frac{x}{6x+25}$$

$$= d(T^2x, S^2x)$$

$$d(STx, TSx) \leq d(T^2x, S^2x).$$

Using [0, 1] for $x \in X$, we conclude that definition (2.2) as follows:

$$d(S^2T^2x, T^2S^2x) \leq d(S^2Tx, TS^2x)$$
$$\leq d(ST^2x, T^2Sx) \leq d(STx, TSx) \leq d(S^2x, T^2x)$$

for any $x \in X$.

We generalized the result of Yogita R Sharma[10], Lohani and Badshah[6], a by using another type of rational expression[7].

3 Main theorem :

Theorem 3.1. If S is a mapping and T is a continuous mapping of complete metric space $\{S, T\}$ is weak commuting pair and the following condition

$$d(S^2T^2x, T^2S^2y) \leq \alpha \frac{[d(T^2x, S^2T^2x)]^3 + [d(S^2y, T^2S^2y)]^3}{[d(T^2x, S^2T^2x)]^2 + [d(S^2y, T^2S^2y)]^2}$$
$$+ \beta \frac{[d(T^2x, S^2y)]^2 + [d(T^2x, T^2S^2y)]^2}{[2d(T^2x, S^2y)]^3 + [3d(T^2x, T^2S^2y)]^3}$$
$$+ \gamma d(T^2x, S^2y) \qquad (3.1.1)$$

for all x, y in X, where $\alpha, \beta \geq 0$ with $2(\alpha + \beta) + \gamma < 1$, then S and T have a unique common fixed point.

Proof. Let x be an arbitrary point X. Define $(S^2T^2)^n x = x_{2n}$ or $T^2(S^2T^2)^n x = x_{2n+1}$,

where $n = 0,1,2,3,\ldots$, by contractive condition (3.1.1),

$$d(x_{2n}, x_{2n+1}) = d\left((S^2T^2)^n x, T^2(S^2T^2)^n x\right)$$
$$= d\left(S^2T^2(S^2T^2)^{n-1} x, T^2S^2\left(T^2(S^2T^2)^{n-1}\right)x\right)$$

$$\leq \alpha \frac{\left\{[d(T^2(S^2T^2)^{n-1}x, S^2T^2(S^2T^2)^{n-1}x)]^3 + [d(S^2T^2(S^2T^2)^{n-1}x, T^2S^2(T^2(S^2T^2)^{n-1})x)]^3\right\}}{\left\{[d(T^2(S^2T^2)^{n-1}x, S^2T^2(S^2T^2)^{n-1}x)]^2 + [d(S^2T^2(S^2T^2)^{n-1}x, T^2S^2(T^2(S^2T^2)^{n-1})x)]^2\right\}}$$

$$+\beta \frac{\left\{ \left[d\left(T^2\left(S^2T^2\right)^{n-1}x, S^2T^2\left(S^2T^2\right)^{n-1}x\right)\right]^2 + \left[d\left(T^2\left(S^2T^2\right)^{n-1}x, T^2S^2\left(T^2\left(S^2T^2\right)^{n-1}\right)x\right)\right]^2 \right\}}{\left\{ \left[2d\left(T^2\left(S^2T^2\right)^{n-1}x, S^2T^2\left(S^2T^2\right)^{n-1}x\right)\right]^3 + \left[3d\left(T^2\left(S^2T^2\right)^{n-1}x, T^2S^2\left(T^2\left(S^2T^2\right)^{n-1}\right)x\right)\right]^3 \right\}}$$

$$+\gamma d\left(T^2\left(S^2T^2\right)^{n-1}x, S^2T^2\left(S^2T^2\right)^{n-1}x\right)$$

$$\leq \alpha \frac{\left[d\left(x_{2n-1}, x_{2n}\right)\right]^3 + \left[d\left(x_{2n}, x_{2n+1}\right)\right]^3}{\left[d\left(x_{2n-1}, x_{2n}\right)\right]^2 + \left[d\left(x_{2n}, x_{2n+1}\right)\right]^2}$$

$$+\beta \frac{\left[d\left(x_{2n-1}, x_{2n}\right)\right]^2 + \left[d\left(x_{2n}, x_{2n+1}\right)\right]^2}{\left[2d\left(x_{2n-1}, x_{2n}\right)\right]^3 + \left[3d\left(x_{2n}, x_{2n+1}\right)\right]^3}$$

$$+\gamma d(x_{2n-1}, x_{2n})$$

$$\leq \alpha\left[d\left(x_{2n-1}, x_{2n}\right) + d\left(x_{2n}, x_{2n+1}\right)\right]$$

$$+\beta[d(x_{2n-1}, x_{2n}) + d(x_{2n}, x_{2n+1})]$$

$$+\gamma d(x_{2n-1}, x_{2n})$$

$$\leq (\alpha+\beta+\gamma)d\left(x_{2n-1}, x_{2n}\right) + (\alpha+\beta)d\left(x_{2n}, x_{2n+1}\right)$$

$$(1-\alpha-\beta)d\left(x_{2n}, x_{2n+1}\right) \leq (\alpha+\beta+\gamma)d\left(x_{2n-1}, x_{2n}\right)$$

$$\Rightarrow d\left(x_{2n}, x_{2n+1}\right) \leq \frac{(\alpha+\beta+\gamma)}{(1-\alpha-\beta)} d\left(x_{2n-1}, x_{2n}\right)$$

$$\Rightarrow d\left(x_{2n}, x_{2n+1}\right) \leq K d\left(x_{2n-1}, x_{2n}\right)$$

where $k = \dfrac{(\alpha+\beta+\gamma)}{(1-\alpha-\beta)}$.

Proceeding in the same manner

$$d\left(x_{2n}, x_{2n+1}\right) \leq k^{2n-1} d\left(x_1, x_2\right).$$

Also $d\left(x_n, x_m\right) \leq \sum_{i=n}^{m} d\left(x_i, x_{i+1}\right)$ for $m > n$.

Since $k < 1$, it follows that the sequence $\{x_n\}$ is Cauchy sequence in the complete metric space X and so it has a limit in X, that is

$$\lim_{n\to\infty} x_{2n} = u = \lim_{n\to\infty} x_{2n+1}$$

and since T is continuous, we have

$$u = \lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} T^2\left(x_{2n}\right) = T^2 u.$$

Further,

$$d\left(x_{2n+1}, S^2 u\right) = d\left(T^2\left(S^2T^2\right)^{n+1}x, S^2 u\right)$$

$$= d\left(T^2\left(S^2T^2\right)^{n+1}x, S^2\left(T^2 u\right)\right)$$

for $u = T^2 u$

$$\leq \alpha \frac{\left[d\left(T^2u, S^2T^2u\right)\right]^3 + \left[d\left(\left(S^2T^2\right)^{n+1}x, T^2\left(S^2T^2\right)^{n+1}x\right)\right]^3}{\left[d\left(T^2u, S^2T^2u\right)\right]^2 + \left[d\left(\left(S^2T^2\right)^{n+1}x, T^2\left(S^2T^2\right)^{n+1}x\right)\right]^2}$$

$$+\beta \frac{\left[d\left(T^2u, (S^2T^2)^{n+1}x\right)\right]^2 + \left[d\left(T^2u, T^2\left(S^2T^2\right)^{n+1}x\right)\right]^2}{\left[2d\left(T^2u, (S^2T^2)^{n+1}x\right)\right]^3 + \left[3d\left(T^2u, T^2\left(S^2T^2\right)^{n+1}x\right)\right]^3}$$

$$+\gamma d(T^2u, (S^2T^2)^{n+1}x)$$

$$\leq \alpha\left[d\left(T^2u, S^2T^2u\right) + d\left(x_{2n+2}, x_{2n+3}\right)\right]$$

$$+\ \beta[d(T^2u, x_{2n+2}) + d(T^2u, x_{2n+3})]$$

$$+\ \gamma d(x_{2n+2}, T^2u)$$

$$\leq \alpha\left[d\left(u, S^2u\right) + d\left(x_{2n+2}, x_{2n+3}\right)\right]$$

$$+\ \beta[d(u, x_{2n+2}) + d(u, x_{2n+3})]$$

$$+\ \gamma d(x_{2n+2}, u)$$

Taking limit as $n \to \infty$, it follows that

$$d\left(u, S^2u\right) \leq 0,$$

which implies

$$d(u,S^2u)=0 \text{ and so } u=S^2u=T^2u.$$

Now consider weak** commutativity of pair $\{S,T\}$ implies that

$S^2T^2u=T^2S^2u, S^2Tu=TS^2u, ST^2u=T^2Su$ and so $S^2Tu=Tu$ and $T^2Su=Su$.

Now,

$$d(u,Su)=d(S^2T^2u,T^2S^2(Su))$$

$$\le \alpha\frac{[d(T^2u,S^2T^2u)]^3+[d(S^2(Su),T^2S^2(Su))]^3}{[d(T^2u,S^2T^2u)]^2+[d(S^2(Su),T^2S^2(Su))]^2}+$$

$$+\beta\frac{[d(T^2u,S^2(Su))]^2+[d(T^2u,T^2S^2(Su))]^2}{[d(T^2u,S^2(Su))]^3+[d(S^2u,T^2S^2(Su))]^3}$$

$$+\gamma d(T^2u,S^2(Su))$$

$$=\alpha\frac{[d(u,u)]^3+[d(Su,Su)]^3}{[d(u,u)]^2+[d(Su,Su)]^2}$$

$$+\beta\frac{[d(u,Su)]^2+[d(u,Su)]^2}{[2d(u,Su)]^3+[3d(u,Su)]^3}+\gamma d(u,Su)$$

$$d(u,Su)\le\beta[d(u,Su)+d(u,Su)]+\gamma d(u,Su)$$

$$(1-2\beta-\gamma)d(u,Su)\le 0.$$

This implies that $(1-2\beta-\gamma)\ne 0$. Hence $d(u,Su)=0$ or $Su=u$.

Similarly, we can show that $Tu=u$. Hence, u is a common fixed point of S and T. Now suppose that v is another common fixed point of S and T. Then

$$d(u,v)=d(S^2T^2u,T^2S^2v)$$

$$\le\alpha\frac{[d(T^2u,S^2T^2u)]^3+[d(S^2v,T^2S^2v)]^3}{[d(T^2u,S^2T^2u)]^2+[d(S^2v,T^2S^2v)]^2}$$

$$+\beta\frac{[d(T^2u,S^2v)]^2+[d(T^2u,T^2S^2v)]^2}{[2d(T^2u,S^2v)]^3+[3d(T^2u,T^2S^2v)]^3}+\gamma d(T^2u,S^2v)$$

$$\le\alpha[d(u,u)+d(v,v)]+\beta[d(u,v)+d(u,v)]+\gamma d(u,v)$$

$$d(u,v)\le\beta d(u,v)+2\gamma d(u,v)$$

$$(1-\beta-2\gamma)d(u,v)\le 0.$$

Since, $(1-\beta-2\gamma)\ne 0$, then $d(u,v)=0$. Thus, it follows that $u=v$. Hence S and T have a unique common fixed point.

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